Math 2013

Midterm # 1

Feb. 3, 2016 Time: 50 minutes Total Marks: 60 10 points for each question

1. Solve the following initial value problem

$$y' = \frac{2x}{y + x^2 y}, \qquad y(0) = -2$$

2. Find the general solution of the following differential equation

$$xy' - 2y = x^3.$$

3. Let

$$\vec{r}(t) = 6\cos(2t)\vec{i} + 6\sin(2t)\vec{j} + 5t\vec{k}.$$

   Find expressions for the unit tangent vector $\vec{T}$ and the curvature κ.

4. Use Green's Theorem to evaluate $\int_C x^4\,dx + xy\,dy$ where C is the triangular curve consisting of the line segments form $(0,0)$ to $(1,0)$ to $(0,1)$.

5. Is the integral

$$\int_C (2xy - 2y^2)dx + (x^2 - 4xy)dy$$

   path independent? Find a potential function that evaluates the integral. Find the value of the integral where C is the path from $(0,0)$ to $(\frac{5\pi}{2}, 1)$ along the curve $y = \sin x$.

6. A force acting on a particle is given by the vector field

$$\vec{F}(x,y) = (2x + y)\vec{i} + (y - x)\vec{j}$$

   Find the work done by $\vec{F}$ in moving a particle along the straight line path from $(0,1)$ to $(1,2)$.

5. $\frac{\partial Q}{\partial x} = 2x - 4y \quad \frac{\partial P}{\partial y} = 2x - 4y$ $\boxed{\text{path indep}}$ ✓ 3

$f = \cancel{x^2y + 2} \boxed{x^2y - 2xy^2 + C}$ 4

$\int_C P\,dx + Q\,dy = f(\frac{5\pi}{2}, 1) - f(0,0) = \frac{25\pi^2}{4} - \frac{5\pi}{\cancel{4}} = \boxed{\frac{25\pi^2 - 5\pi}{4}}$ 3

6. $\int_C \vec{F} \cdot d\vec{r} = \int_C (2x+y)\,dx + (y-x)\,dy = \int_0^1 (2t+t+1)\,dt + (t+1-t)\,dt$

$x = t \quad dx = dt$
$y = t+1 \quad dy = dt$

$= \int_0^1 (3t+2)\,dt = \frac{3}{2} + 2 = \boxed{7/2}$

1. $yy' = \frac{2x}{1+x^2} \rightarrow \int y\,dy = \int \frac{2x\,dx}{1+x^2} \rightarrow \frac{y^2}{2} = \ln(1+x^2) + C$

$y = \pm\sqrt{2\ln(1+x^2) + C} \qquad y(0) = -2$

$-2 = \pm\sqrt{2\ln 1 + C} \qquad C = 4$

$\boxed{y = -\sqrt{2\ln(1+x^2) + 4}}$

2. $y' - \frac{2}{x}y = x^2 \qquad e^{\int p\,dx} = e^{-\int \frac{2}{x}dx} = e^{-2\ln x} = x^{-2}$

$(x^{-2}y)' = 1 \rightarrow x^{-2}y = x + C \qquad \boxed{y = x^3 + Cx^2}$

3. $\vec{r}\,' = \langle -12\sin 2t, 12\cos 2t, 5\rangle$

$|\vec{r}\,'| = \sqrt{144\sin^2 2t + 144\cos^2 2t + 25} = \sqrt{169} = 13$

$\vec{T} = \frac{1}{13}\langle -12\sin 2t, 12\cos 2t, 5\rangle$

$K = \frac{|T'|}{|r'|} = \frac{\frac{1}{13}|\langle -24\cos 2t, -24\sin 2t, 0\rangle|}{13} = \frac{\frac{24}{13}}{13} = \frac{24}{169}$

$= \frac{|r'\times r''|}{|r'|^3} = \frac{\begin{vmatrix} i & j & k \\ -12\sin 2t & 12\cos 2t & 5 \\ -24\cos 2t & -24\sin 2t & 0 \end{vmatrix}}{13^3} = \cancel{+\langle +60\sin 2t, 120\cos 2t,}$

$= \frac{+24}{13^3}\begin{vmatrix} i & j & k \\ 12\sin 2t & 12\cos 2t & 5 \\ -\cos 2t & -\sin 2t & 0 \end{vmatrix} = \frac{24}{13^3}\langle -5\sin 2t, 5\cos 2t, +12\sin^2 2t + 12\cos^2 2t\rangle$

$= \frac{24}{13^3}|\sqrt{5^2+12^2}| = \frac{24}{13^2}$

4,  $\frac{\partial Q}{\partial x} = x$ $\frac{\partial P}{\partial y} = 0$

$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint (x - 0)\, dA$

$\int_0^1 \int_0^{1-x} x\, dy\, dx = \int_0^1 (x(1-x))\, dx = \int_0^1 \left(x - \frac{x^2}{2}\right) dx = \frac{1}{2} - \frac{1}{3} = \boxed{\frac{1}{6}}$

$\int_0^1 \int_0^{1-y} x\, dx\, dy = \int_0^1 \frac{(1-y)^2}{2}\, dy = \int_0^1 \frac{1-2y+y^2}{2}\, dy = \frac{1}{2} - \frac{1}{2} + \boxed{\frac{1}{6}}$