Math 2013

Midterm # 2

Mar. 2, 2016 Time: 50 minutes Total Marks: 40 10 points for each question

Recall:

$$\sin^2(x) = \frac{1-\cos(2x)}{2}$$
$$\cos^2(x) = \frac{1+\cos(2x)}{2}$$

1. Let $\vec{F} = y^2\vec{i} + (2xy + e^{3z})\vec{j} + 3ye^{3z}\vec{k}$.
   (a) Compute $\nabla \cdot \vec{F}$.
   (b) Find a potential function f such that $\nabla f = \vec{F}$.
   (c) Evalute the line integral $\int_C \vec{F} \cdot d\vec{r}$.
   Where C is parametrized by $\vec{r} = \cos(t)\vec{i} + \sin(t)\vec{j} + t\vec{k}$ for $0 \le t \le \pi/2$.
2. Use the Divergence Theorem to evaluate $\iint_S \vec{F} \cdot \vec{n}\, dS$ where
   $$\vec{F} = xz\vec{i} + e^{xz^2}\vec{j} + (\sin(xy) + z^2/2)\vec{k}$$
   and S is the surface which is the boundary of the region in space D bounded by the parabolic cylinder $z = 1 - y^2$ and the planes $x = 0$ and $x = 4$ and the xy-plane.
3. Evalute $\int_C x^4\, dx + xy\, dy$ where C is the triangular curve consisting of the line segments from $(0,0)$ to $(1,0)$ to $(0,1)$.
4. Use Stoke's Theorem to evaluate $\iint_S \mathrm{curl}\vec{\mathrm{F}} \cdot \vec{\mathrm{n}}\, \mathrm{dS}$ where $\vec{F} = 2y\vec{i} - x\vec{j} + z\vec{k}$ and S is the hemisphere $x^2 + y^2 + z^2 = 4$ and $z \ge 0$ with outward pointing normal vector.

1. a) $\nabla \cdot \vec{F} = 0 + 2x + 9y\, e^{3z}$

b) $f = xy^2 + y e^{3z}$

c) $\int_C \nabla \vec{F} \cdot d\vec{r} = \vec{F}(\vec{r}(\pi/2)) - \vec{F}(\vec{r}(0))$
$= e^{3\pi/2} - 0$

2. $\iint_S \vec{F} \cdot \vec{n}\, dS = \iiint_D \operatorname{div} \vec{F}\, dV = \int_0^4 \int_{-1}^{1} \int_0^{1-y^2} 2z\, dz\, dy\, dx$

$= 2 \cdot 4 \int_{-1}^{1} z^2 \Big]_0^{(1-y^2)} dy = 4 \cdot 2^2 \int_0^1 (1-y^2)^2\, dy = 2 \cdot 8 \left(y - \frac{2y^3}{3} + \frac{y^5}{5}\right)\Big]_0^1$

$= \cancel{16/3}\ 64/15$

3. $\int_C x^4 dx + xy\, dy = \iint_\Delta y\, dA = \int_0^1 \int_0^{1-x} y\, dy\, dx = \int_0^1 \frac{y^2}{2}\Big]_0^{1-x} dx$

$= \int_0^1 \frac{(1-x)^2}{2} dx = \frac{1}{2}\left(x - x^2 + \frac{x^3}{3}\right)\Big]_0^1 = 1/6$

4. $\iint \operatorname{curl} \vec{F} \cdot \vec{n}\, dS = \oint_C \vec{F} \cdot d\vec{r}$ | $\vec{r} = \langle 2\cos\theta, 2\sin\theta, 0 \rangle$
$\vec{r}\,' = \langle -2\sin\theta, 2\cos\theta, 0 \rangle$

$= \int_0^{2\pi} \langle 4\sin\theta, -\cos\theta, 0 \rangle \cdot \langle -2\sin\theta, 2\cos\theta, 0 \rangle\, d\theta$

$= \int_0^{2\pi} (-8\sin^2\theta - 4\cos^2\theta)\, d\theta = \int_0^{2\pi} (-4(1-\cos 2\theta) - 2(1+\cos 2\theta))\, d\theta$

$= -4 \cdot 2\pi - 2 \cdot 2\pi = -12\pi$