10 pts each. /100

Math 2203 Assignment #5 solutions

3.2

1. a) $\gcd(561, 330) = 33 = 3\cdot 561 - 5\cdot 330$
   b) $\gcd(3542, 276) = 46 = 13\cdot 276 - 1\cdot 3542$
   c) $\gcd(2145, 663) = 39 = 13\cdot 663 - 4\cdot 2145$
   d) $\gcd(509, 132) = 27\cdot 132 - 7\cdot 509$
   e) $\gcd(10166, 1932) = 46 = 100\cdot 1932 - 19\cdot 10166$
   f) $\gcd(1012, 295) = 1 = 247\cdot 295 - 72\cdot 1012$

3. $sa + tb = 21$, $ua + vb = 10 \Rightarrow (sa+tb) - 2(ua+vb) = 1$
$\Rightarrow (s-2u)a + (t-2v)b = 1 \Rightarrow \gcd(a,b) = 1$ by Ex 6 below

Or: Let $d = \gcd(a,b) \Rightarrow d|a$ & $d|b \Rightarrow d | sa+tb = 21$
& $d | ua+vb = 10 \Rightarrow d | \gcd(21, 10) = 1 \Rightarrow d = 1$

4. $2\times(5 \text{ cents}) - 2\times(3 \text{ cents}) = 4$ cents
Many possible solns, for example:
$2m\times(3\text{ cents}) - m\times(5\text{ cents}) = m$ for any m,

6 Given $a, b \in \mathbb{Z}$, non-zero, ~~then~~
a) $\exists s,t \in \mathbb{Z}$ s.t. $sa + tb = \gcd(a,b)$ by Euclid Alg.
& $\gcd(a,b) > 0$ by defn, so $\gcd(a,b) \in L^+$.
b) $z \in L \Rightarrow \exists s,t$ so $z = sa + bt$. Now $\gcd(a,b) | a$
& $\gcd(a,b) | b \Rightarrow \gcd(a,b) | sa + bt = z$.
So $\gcd(a,b) | z$ for any $z \in L$
c) Let $z \in L^+ \Rightarrow \gcd(a,b) | z \Rightarrow |\gcd(a,b)| \le |z|$
both are > 0 so $\gcd(a,b) \le z$, $\gcd(a,b) \in L^+$
so $\gcd(a,b)$ is smallest element of L^+

10 To show: $\gcd(k, k+n) = \gcd(k, n)$
Run Euclidean Alg: $k+n = q_1 k + r_1$ $\quad n = (q_1 - 1)k + r_1$
$k = q_2 r_1 + r_2$ etc $\quad k = q_2 r_1 + r_2$ etc Same.
So get $\gcd(k+n, k) = \gcd(k, r_1) = \gcd(k, n)$

12) Another way: $L^+ = \{ s(k+n) + tk \mid s,t \in \mathbb{Z}, s(k+n)+tk > 0 \}$
$M^+ = \{ sk + tn \mid sk+tn > 0, s,t \in \mathbb{Z} \}$

Claim $L^+ = M^+$. proof) Let $x \in L^+ \Rightarrow x = s(k+n) + tk, x > 0$
$\Rightarrow x = (s+t)k + sn, x > 0 \Rightarrow x \in M^+$

Similarly $x \in M^+ \Rightarrow x \in L^+$. So $L^+ = M^+$. By Ex 6.
$\gcd(k+n, k) = \min L^+ = \min M^+ = \gcd(k,n)$

3.3) 1. a) $a = 2^4 \cdot 5^2 \cdot 11$ $\quad$ $\gcd = 2^2$
$b = 7 \cdot 2^2 \cdot 3^3$ $\quad$ $\operatorname{lcm} = 2^4 3^3 5^2 \cdot 7 \cdot 11$

b) $a = 2^2 \cdot 11 \cdot 13$ $\quad$ $\gcd = 1$
$b = 3^2 \cdot 5^2$ $\quad$ $\operatorname{lcm} = 2^2 3^2 5^2 \cdot 11 \cdot 13$

c) $a = 5 \cdot 7^2 \cdot 11$ $\quad$ $\gcd = 5 \cdot 7$
$b = 2^2 \cdot 5 \cdot 7 \cdot 13$ $\quad$ $\operatorname{lcm} = 2^2 \cdot 5 \cdot 7^2 \cdot 11 \cdot 13$

d) $a = 2^4 3^2 7\ 13^2$ $\quad$ $\gcd = 2^2 \cdot 3$
$b = 2^2 3 \cdot 11^2 \cdot 17^2$ $\quad$ $\operatorname{lcm} = 2^4 \cdot 3^2 \cdot 7 \cdot 11^2 \cdot 13^2 17^2$

e) $a = 19 \cdot 23^2$ $\quad$ $\gcd = 23^2$
$b = 3^2 \cdot 17 \cdot 23^2$ $\quad$ $\operatorname{lcm} = 3^2 \cdot 17 \cdot 19 \cdot 23^2$

3. Let $e, f \in \mathbb{Z}$. Note that $\max(e,f) + \min(e,f) = e+f$
since if $e \geq f$ or if $e \leq f$ it's true.
Let $m = p_1^{e_1} \cdots p_k^{e_k}$, $n = p_1^{f_1} \cdots p_k^{f_k}$ $\quad e_i, f_i \geq 0$
prime factorization
$\gcd(m,n) \operatorname{lcm}(m,n) = p_1^{\min(e_1,f_1)} \cdots p_k^{\min(e_k,f_k)} p_1^{\max(e_1,f_1)} \cdots p_k^{\max(e_k,f_k)}$
$= p_1^{\min(e_1,f_1)+\max(e_1,f_1)} \cdots p_k^{\min(e_k,f_k)+\max(e_k,f_k)}$
$= p_1^{e_1+f_1} \cdots p_k^{e_k+f_k} = p_1^{e_1} \cdots p_k^{e_k} p_1^{f_1} \cdots p_k^{f_k} = mn$

11. Write prime factorization $n = p_1^{e_1} \cdots p_k^{e_k}$
If all $e_i = 2f_i$ are even then $\sqrt{n} = p_1^{f_1} \cdots p_k^{f_k} \in \mathbb{Z}$
Else some e_i is odd. Then $\sqrt{n} = p/q$ so $q^2 n = p^2$, $p,q \in \mathbb{Z}$
Exponent of p_i on left is odd, Exponent of p_i on right is even. Contradicts Fun Thm Arith. So $\sqrt{n} \notin \mathbb{Q}$

16. Multiples of 5 : $1000/5 = 200$
Multiples of 25 : $1000/25 = 40$
Multiples of 125 : $1000/125 = 8$
Multiples of 625 : $1000/625 = 1.6$

$$\overline{249}$$

$\boxed{249}$

21. Several ways to do this! One way

$(a, b, a+b-3)$: $a^2 + b^2 = c^2 = (a+b-3)^2$
$= a^2 + b^2 + 9 + 2ab - 6a - 6b$

$9 = 2ab - 6a - 6b$
odd = even contradiction.