Principal coactions of discrete groups and K-theory of noncommutative line bundles

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Hopf-Galois theory

A comodule algebra P over a Hopf algebra H with bijective antipode is called *principal* if the coaction δ of H is Galois and P is H-equivariantly projective (faithfully flat) over the coaction-invariant subalgebra P^{coH}.

Here the Galois condition means that the canonical map

$$P \underset{P^{coH}}{\otimes} P \ni p \underset{P^{coH}}{\otimes} q \longmapsto (p \otimes 1)\delta(q) \in P \otimes H$$

is an isomorphism, and the equivariant projectivity means that there exists a left P^{coH}-linear right H-colinear splitting of the multiplication map

$$P^{coH} \otimes P \longrightarrow P.$$

Galois coverings

Theorem: *Let $p : X \to M$ be a covering map, where X and M are locally path connected. Then the following are equivalent:*

1. $\exists\, x_0 \in X : p_*(\pi_1(X, x_0))$ *is a* normal *subgroup of* $\pi_1(M, p(x_0))$,

2. $\exists$ *a discrete group* Γ *such that* $p : X \to M$ *is a* Γ-principal *bundle.*

Theorem: *Let $K \subseteq L \subseteq N$ be an extension of fields, where N is Galois over K. Then the following are equivalent:*

1. $Gal(N/L)$ *is a* normal *subgroup of* $Gal(N/K)$,

2. $K \subseteq L$ *is* Galois.

Actions of compact quantum groups

Definition (S.L.Woronowicz): A *compact quantum group* is a pair $(\bar{H}, \Delta)$, where $\bar{H}$ is a unital C^*-algebra and $\Delta : \bar{H} \to \bar{H} \otimes_{\min} \bar{H}$ is a coassociative $*$-homomorphism (called comultiplication) such that the two-sided cancellation property holds:

$$\{(a \otimes 1)\Delta(b) \mid a, b \in \bar{H}\}^{\mathrm{cls}} = \bar{H} \underset{\min}{\otimes} \bar{H} = \{\Delta(a)(1 \otimes b) \mid a, b \in \bar{H}\}^{\mathrm{cls}}.$$

Definition: Let A be a C^*-algebra and $\delta : A \to A \otimes_{\min} \bar{H}$ an injective $*$-homomorphism. We call δ a *coaction* if

1. $(\delta \otimes \mathrm{id}) \circ \delta = (\mathrm{id} \otimes \Delta) \circ \delta$ (coassociativity)
2. $\{\delta(a)(1 \otimes h) \mid a \in A,\, h \in \bar{H}\}^{\mathrm{cls}} = A \underset{\min}{\otimes} \bar{H}$ (counitality)

The Peter-Weyl subalgebra

Let $\bar{H}$ be the C^*-algebra of a compact quantum group and H its dense Hopf $*$-subalgebra spanned by the matrix coefficients of the irreducible unitary corepresentations. Let A be a unital C^*-algebra and let $\delta : A \to A \otimes_{\min} \bar{H}$ be a coaction. We define the Peter-Weyl subalgebra of A as:

$$PW_H(A) := \{\, a \in A \,|\, \delta(a) \in A \otimes H \,\}.$$

Properties:

- $PW_H(A)$ is an H-comodule algebra,
- $A^{co\bar{H}} = PW_H(A)^{coH}$ (invariants are the same),
- $PW_H(A)$ is a dense $*$-subalgebra of A,
- PW_H is a functor commuting with equivariant pullbacks.

Compact principal bundles

The compactness of G implies that its action is proper, and freeness is equivalent to the bijectivity of the map

$$\widetilde{F} : X \times G \ni (x, g) \longmapsto (x, xg) \in X \underset{X/G}{\times} X.$$

Theorem (P.F.Baum, P.M.H.): *Let G be a compact group and X a compact Hausdorff G-bundle. Then X is a principal bundle if and only if*

$$\widetilde{F}^* : PW_G(X) \underset{C(X/G)}{\otimes} PW_G(X) \longrightarrow PW_G(X) \otimes \mathcal{O}(G)$$

is an isomorhism.

Associated projective modules

If M is a right comodule over a coalgebra C and N is a left C-comodule, then we define their *cotensor product* as

$$M \overset{C}{\Box} N := \{t \in M \otimes N \mid (\Delta_M \otimes \mathrm{id})(t) = (\mathrm{id} \otimes {}_N\Delta)(t)\}.$$

In particular, for a right H-comodule algebra P and a left H-comodule V, we observe that $P\Box^H V$ is a left P^{coH}-module in a natural way. If P is a principal comodule algebra and $\dim V < \infty$, then $P\Box^H V$ is *finitely generated projective*. Furthermore, if $\dim V = 1$, then any coaction on V is given by a grouplike $\gamma \in H$, and we obtain:

$$P \overset{C}{\Box} V = \{p \in P \mid \delta(p) = p \otimes \gamma\} =: P_\gamma.$$

Characters and fibres

Lemma: *Let* P *be a principal comodule algebra over a Hopf algebra* H*, and* $P \xrightarrow{\chi} k$ *be a character making* k *a module over* P *and any of its subalgebras. Then, for any left comodule* V*,*

$$\boxed{k \underset{P^{coH}}{\otimes} (P \overset{H}{\Box} V) \cong V}.$$

Proof: Using the right flatness of P implied by its principality, we obtain

$$\begin{aligned} k \underset{P^{coH}}{\otimes} (P \overset{H}{\Box} V) &\cong k \underset{P}{\otimes} (P \underset{P^{coH}}{\otimes} (P \overset{H}{\Box} V)) \cong k \underset{P}{\otimes} ((P \underset{P^{coH}}{\otimes} P) \overset{H}{\Box} V) \\ &\cong k \underset{P}{\otimes} ((P \otimes H) \overset{H}{\Box} V) \cong k \underset{P}{\otimes} (P \otimes (H \overset{H}{\Box} V)) \\ &\cong k \underset{P}{\otimes} (P \otimes V) \cong (k \underset{P}{\otimes} P) \otimes V \cong k \otimes V \cong V. \end{aligned}$$

Principal coactions of discrete groups

Let P be a comodule algebra over a group ring $H = k\Gamma$. This is equivalent to P being graded by Γ:

$$P = \bigoplus_{\gamma\in\Gamma} P_\gamma, \quad P_\gamma := \{p \in P \mid \delta(p) = p \otimes \gamma\},$$

$$P_\gamma P_{\gamma'} \subseteq P_{\gamma\gamma'}, \quad \forall\, \gamma,\, \gamma' \in \Gamma.$$

Theorem (K.H.Ulbrich): *The coaction of $k\Gamma$ on P is principal (Galois) if and only if P is* strongly graded *by Γ, i.e.,*

$$P_\gamma P_{\gamma'} = P_{\gamma\gamma'},\ \forall\, \gamma,\, \gamma' \in \Gamma.$$

Stable triviality criterion

Theorem (P.F.Baum, P.M.H.): *Let* $\delta : P \to P \otimes k\Gamma$ *be a principal coaction, and* $P_\gamma := \{p \in P \mid \delta(p) = p \otimes \gamma\}$. *Then* P_γ *is* stably free *over* $P^{co\,k\Gamma}$ *if and only if there exist matrices* $T \in M_{m,n+1}(P)$ *and* $T^{-1} \in M_{n+1,m}(P)$ *such that:*

1. *the first column of* T *has entries from* P_γ *and all other columns have entries from* $P^{co\,k\Gamma}$*,*

2. *the first row of* T^{-1} *has entries from* $P_{\gamma^{-1}}$ *and all other rows have entries from* $P^{co\,k\Gamma}$*,*

3. $T^{-1}T = I_{n+1}$ *and* $TT^{-1} = I_m$*, where* I_k *is the identity matrix of size* k*.*

Corollary: *If* P *admits a character, then* $m = n + 1$ *and the above matrices are square. In particular,* T *defines a class in* $K_1(P)$*.*

Proof of the criterion

Let e be the neutral element of Γ and $P_\gamma \oplus P_e^n \cong P_e^m$. Then there exists $T \in M_{m,n+1}(P)$ whose rows form a basis of $P_\gamma \oplus P_e^n$. By construction, the first column of T has entries from P_γ and all other columns have entries from P_e. By Ulbrich's theorem,

$$\exists\, x_i \in P_\gamma,\ y_i \in P_{\gamma^{-1}},\ i = 0, \cdots, N \ :\ \sum_{i=0}^{N} y_i x_i = 1.$$

Define the following diagonal matrices in $M_{n+1}(P)$:

$$X_0 := \operatorname{diag}(x_0, 1, \cdots, 1), \qquad X_{i>0} := \operatorname{diag}(x_i, 0, \cdots, 0),$$
$$Y_0 := \operatorname{diag}(y_0, 1, \cdots, 1), \qquad Y_{i>0} := \operatorname{diag}(y_i, 0, \cdots, 0).$$

Since the rows of T span $P_\gamma \oplus P_e^n$, we have

$$\forall\, i = 0, \cdots, N\ \exists\, T_i \in M_{n+1,m}(P_e)\ :\ X_i = T_i T.$$

On the other hand, define $T^{-1} := \sum_{i=0}^{N} Y_i T_i \in M_{n+1,m}(P)$. The entries of the first row of T^{-1} are from $P_{\gamma^{-1}}$. Furthermore,

$$T^{-1}T = \sum_{i=0}^{N} Y_i T_i T = \sum_{i=0}^{N} Y_i X_i = I_{n+1}.$$

Hence $(TT^{-1})T = T$. Now, taking advantage of $TT^{-1} \in M_m(P_e)$, we conclude that $TT^{-1} = I_m$ by the linear independence of the rows of T.

Conversely, assume that there exist matrices T and T^{-1} satisfying the conditions of the theorem. Then $vT^{-1} \in P_e^m$ and $v = (vT^{-1})T$ for any $v \in P_\gamma \oplus P_e^n$. Hence the rows of T span $P_\gamma \oplus P_e^n$. Also, if $w \in P_e^m$ and $wT = 0$, then $w = wTT^{-1} = 0$, so that the rows of T are linearly independent over P_e.

Finally, if P admits a character χ, then $m = n+1$ because χ applied componentwise to T yields a matrix implementing an isomorphism $k^m \cong k^{n+1}$.

Strong connections

A *strong connection* ℓ on a right H-comodule algebra P is a unital linear map $\ell : H \to P \otimes P$ satisfying

$$(\mathrm{id}\otimes\Delta_P)\circ\ell = (\ell\otimes\mathrm{id})\circ\Delta, \quad ({}_P\Delta\otimes\mathrm{id})\circ\ell = (\mathrm{id}_H\otimes\ell)\circ\Delta\,, \quad \widetilde{can}\circ\ell = 1\otimes\mathrm{id}.$$

$$\begin{array}{ccc} H & \xrightarrow{\ \ell\ } & P\otimes P \\ {\scriptstyle 1\otimes\mathrm{id}}\downarrow & \swarrow{\scriptstyle \widetilde{can}} & \downarrow{\scriptstyle \text{canonical surjection}} \\ P\otimes H & \underset{can}{\longleftarrow} & P\otimes_B P\,. \end{array}$$

We use the Heyneman-Sweedler-type notation: $h \mapsto \ell(h)^{\langle 1\rangle} \otimes \ell(h)^{\langle 2\rangle}$ (summation suppressed). *A comodule algebra is principal if and only if it admits a strong connection.*

The Chern-Galois character

1. Associated module construction $(V \stackrel{\varphi}{\to} H \otimes V) \mapsto P\square_H V$ defines a homomorphism from the Grothendieck group $R_f(H)$ of all isomorphism classes of finite-dimensional corepresentations of H into $K_0(P^{coH})$.

2. Combining this homomorphism with the Chern character, $\forall\ n \in \mathbb{N}$: $K_0(P^{coH}) \to HC_{2n}(P^{coH})$, yields a homomorphism

$$chg_{2n} : R_f(H) \longrightarrow HC_{2n}(P^{coH}).$$

   called the *Chern-Galois character*.

Theorem (T.Brzeziński, P.M.H.): *Let* $c^\varphi := \sum_{i=1}^{\dim \varphi} e_{ii}$ *be the character of* φ. *Then the Chern-Galois character has the following explicit form* $(-1)^n chg_{2n}([\varphi]) =:$

$$[\ell(c^\varphi_{(2n+1)})^{<2>}\ell(c^\varphi_{(1)})^{<1>}\otimes\ell(c^\varphi_{(1)})^{<2>}\ell(c^\varphi_{(2)})^{<1>}\otimes\ldots\otimes\ell(c^\varphi_{(2n)})^{<2>}\ell(c^\varphi_{(2n+1)})^{<1>}].$$

Quantum Hopf fibration

Hopf $*$-algebra $\mathcal{O}(SU_q(2))$, $q \in]0,1[$, is generated by α and β satisfying:

$$q\alpha\beta = \beta\alpha, \quad q\alpha\beta^* = \beta^*\alpha, \quad \beta\beta^* = \beta^*\beta, \quad \alpha^*\alpha + q^2\beta^*\beta = 1, \quad \alpha\alpha^* + \beta\beta^* = 1.$$

The comultiplication Δ is defined by the formula:

$$\Delta \begin{pmatrix} \alpha & \beta \\ -q\beta^* & \alpha^* \end{pmatrix} = \begin{pmatrix} \alpha \otimes 1 & \beta \otimes 1 \\ -q\beta^* \otimes 1 & \alpha^* \otimes 1 \end{pmatrix} \begin{pmatrix} 1 \otimes \alpha & 1 \otimes \beta \\ 1 \otimes (-q\beta^*) & 1 \otimes \alpha^* \end{pmatrix}$$

The *standard quantum Hopf fibration* of $SU_q(2)$ is given by the coaction

$$\delta := (\mathrm{id} \otimes \pi) \circ \Delta : \mathcal{O}(SU_q(2)) \longrightarrow \mathcal{O}(SU_q(2)) \otimes \mathcal{O}(U(1)),$$

$$\pi : \mathcal{O}(SU_q(2)) \to \mathcal{O}(U(1)) = \mathbb{C}[u, u^*] = \mathbb{C}\mathbb{Z}, \quad \pi(\alpha) = u, \quad \pi(\beta) = 0.$$

Index pairing for quantum Hopf line bundles

For natural numbers $0 \leq l \leq \nu$, we define the x-shifted binomial $\left[\begin{smallmatrix} \nu \\ l \end{smallmatrix}\right]_x$ by the equality $\sum_{l=0}^{\nu} \left[\begin{smallmatrix} \nu \\ l \end{smallmatrix}\right]_x t^l := \prod_{l=0}^{\nu-1}(1 + x^l t)$, and the x-binomial as

$$\begin{pmatrix} \nu \\ l \end{pmatrix}_x := \frac{(x-1)\ldots(x^n-1)}{(x-1)\ldots(x^k-1)(x-1)\ldots(x^{n-k}-1)}.$$

Theorem (P.M.H.): *The index pairing of any quantum Hopf line bundle $\mathcal{O}(SU_q(2))_\mu$ with the non-trivial generator F^1 of the K-homology of the standard Podleś sphere coincides with minus its winding number, i.e., $\langle [F^1], [\mathcal{O}(SU_q(2))_\mu] \rangle = -\mu$, $\forall\ \mu \in \mathbb{Z}$.*

Proof: $\langle [F^1], [\mathcal{O}(SU_q(2))_n] \rangle = (\mathrm{Tr}_{F^1} \circ chg_0)(\varphi_n) = \mathrm{Tr}_{F^1}(\ell(u^n)^{<2>} \ell(u^n)^{<1>})$

$$= \sum_{m=1}^{n} (1 - q^{2m})^{-1} (-1)^m \sum_{k=0}^{m} \binom{n}{k}_{q^2} q^{-2k(n-k-1)} (-1)^k \left[\begin{smallmatrix} n-k \\ m-k \end{smallmatrix}\right]_{q^{-2}} = -n.$$

The case $\mu = -n$ is computed by a similar expression.

The antipodal action on quantum spheres

Let $0 < q \leq 1$. The C*-algebra $C(S_q^{2n+1})$ of an odd quantum sphere is the universal C*-algebra generated by $z_0, z_1, \ldots, z_n$ subject to the relations:

$$z_j z_i = q z_i z_j \quad \text{for } i < j, \qquad z_j^* z_i = q z_i z_j^* \quad \text{for } i < j,$$
$$[z_i^*, z_i] = (1 - q^2) \textstyle\sum_{j=i+1}^n z_j z_j^* \quad \text{for all } i, \qquad \textstyle\sum_{j=0}^n z_j z_j^* = 1.$$

The C*-algebras of even quantum spheres are $C(S_q^{2n}) := C(S_q^{2n+1})/\langle z_n - z_n^* \rangle$, $n \in \mathbb{N}$. We also have $C(S_q^{2n+1}) = C(S_q^{2n+2})/\langle z_{n+1} \rangle$, $n \in \mathbb{N}$.

Furthermore, the formula $\alpha_{-1}(z_i) := -z_i$ gives the antipodal $\mathbb{Z}/2\mathbb{Z}$-action on all these C*-algebras. It is compatible with the quotient maps, so that we have a sequence of $\mathbb{Z}/2\mathbb{Z}$-equivariant epimorphisms of C*-algebras:

$$\cdots \longrightarrow C(S_q^{n+1}) \longrightarrow C(S_q^n) \longrightarrow \cdots \longrightarrow C(S_q^2) \longrightarrow C(S^1) \longrightarrow C(\mathbb{Z}/2\mathbb{Z}).$$

Quantum real projective spaces

The $\mathbb{Z}/2\mathbb{Z}$-action decomposes our quantum sphere C*-algebras into even and odd parts: $C(S_q^n) = C(S_q^n)^{\mathbb{Z}/2\mathbb{Z}} \oplus \{p \in C(S_q^n) \mid \alpha_{-1}(p) = -p\}$. The former defines a quantum real projective space $C(\mathbb{R}P_q^n) := C(S_q^n)^{\mathbb{Z}/2\mathbb{Z}}$ and the latter its tautological line bundle

$$\boxed{L_n := \{p \in C(S_q^n) \mid \alpha_{-1}(p) = -p\}}.$$

To use the framework of principal comodule algebras, note that the above $\mathbb{Z}/2\mathbb{Z}$-action is equivalent to the right coaction of $C(\mathbb{Z}/2\mathbb{Z})$ given by the formula $\delta(z_i) := z_i \otimes \gamma$, $\gamma(\pm 1) := \pm 1$. Since $\ell(\gamma) := \sum_{j=0}^{n} z_j \otimes z_j^*$ defines a strong connection for any $n \in \mathbb{N}$, all these comodule algebras are principal and the associated modules $L_n = C(S_q^n)_\gamma \cong C(S_q^n)\Box_{C(\mathbb{Z}/2\mathbb{Z})}\mathbb{C}$ are finitely generated projective.

Stable non-triviality of the tautological line bundles

Theorem (P.F.Baum, P.M.H.): *For any $n > 1$, the left $C(\mathbb{R}P_q^n)$-module L_n is not stably free.*

Proof: Suppose that L_n is stably free. Then, by the principality of $C(S_q^n)$ and the stable-freeness criterion, there exists a matrix T_n satisfying the conditions of the criterion. The matrix is square because $C(S_q^n)$ admits a character. Composing quotient maps, we obtain a $\mathbb{Z}/2\mathbb{Z}$-equivariant C*-epimorphism

$$f : C(S_q^n) \longrightarrow C(S_q^2) \cong \{(t_1, t_2) \in \mathcal{T}^2 \mid \sigma(t_1) = \sigma(t_2)\}.$$

Here $\mathcal{T}$ is the Toeplitz algebra and $\sigma : \mathcal{T} \to C(S^1)$ is the symbol map. Applying f component-wise to the matrix T_n yields a matrix T_2 that, due to the $\mathbb{Z}/2\mathbb{Z}$-equivariance of f, also satisfies the conditions of the criterion.

Furthermore, using the pullback structure of $C(S_q^2)$, one can check that

$$C(\mathbb{R}P_q^2) \cong \{p \in \mathcal{T} \mid \alpha_{-1}(\sigma(p)) = \sigma(p)\}\,,\ L_2 \cong \{p \in \mathcal{T} \mid \alpha_{-1}(\sigma(p)) = -\sigma(p)\}\,.$$

With these identifications, T_2 becomes an invertible matrix with entries in $\mathcal{T}$ such that its first column has entries in $\{p \in \mathcal{T} \mid \alpha_{-1}(\sigma(p)) = -\sigma(p)\}$ and all other columns have entries in $\{p \in \mathcal{T} \mid \alpha_{-1}(\sigma(p)) = \sigma(p)\}$. Therefore

$$\det\left((\mathrm{id} \otimes \sigma)(T_2)\right)(-\lambda) = -\det\left((\mathrm{id} \otimes \sigma)(T_2)\right)(\lambda).$$

Hence the winding number of $\det\left((\mathrm{id} \otimes \sigma)(T_2)\right)$ is *odd*. On the other hand, it is equal to

$$[(\mathrm{id} \otimes \sigma)(T_2)]_{K_1(C(S^1))} = \sigma_*[T_2]_{K_1(\mathcal{T})} = 0.$$

This yields the desired contradiction proving the theorem.