Noncommutative Surfaces

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Noncommutative Geometry Summer School
IUT

Outline

Noncommutative Algebras

Familiar Examples

Noncommutative Geometry — Functions do not commute

$$yx \neq xy$$

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- **Quaternions** $\mathbb{H} = \mathbb{R}1 \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$
  $i^2 = j^2 = k^2 = ijk = -1$

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  $i^2 = j^2 = k^2 = ijk = -1$
  $\mathbb{H} = \mathbb{R}\langle i, j\rangle/(i^2 = j^2 = -1, ji = -ij)\ k = ij$

Matrices with generators and relations

$$\mathbb{C}^{n\times n} \simeq \mathbb{C}\langle x, y\rangle/(x^n = 1, y^n = 1, yx = \zeta xy)$$
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$$x = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 0 & 0 \end{pmatrix} \qquad y = \begin{pmatrix} \zeta & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \zeta^n \end{pmatrix}$$

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Properties in common with polynomial functions $\mathbb{C}[x, y]$

surface basis $\{x^i y^j\}$

irreducible domains $fg = 0 \Rightarrow f = 0$ or $g = 0$

smooth good homological algebra properties

Sklyanin algebra

Sklyanin algebra Artin, Tate, and Van den Bergh
Noncommutative $\mathbb{CP}^2$

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Has similar good properties.

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elliptic curve E, an automorphism $\sigma \in \operatorname{Aut} E$, and Λ degree three line bundle on E.

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Applications Noncommutative algebraic geometry applied to commutative algebraic geometry, physics.

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Varieties of modules of dimension n.

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$\mathrm{BSV}(\mathcal{A}) = \{\mathcal{A} \twoheadrightarrow \mathbb{C}^n : \text{right module maps}\}$
$= \{(u, v), [x, y, z] \in \mathbb{C}^2 \times \mathbb{CP}^2 : ux^2 + vy^2 + z^2 = 0\}$

Matrix algebra bundles

Z is an algebraic surface over $\mathbb{C}$.
V is a rank n vector bundle over Z.
$\mathcal{A} = \mathcal{E}nd\, V \simeq V \otimes V^*$ is a noncommutative surface.
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Morita equivalence $A \overset{M}{\simeq} B \Leftrightarrow \mathrm{Mod}\, A \simeq \mathrm{Mod}\, B$

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Have degenerations of $\mathbb{C}^{n \times n}$ at
points in X/G with nontrivial stabilizers

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3. $A \otimes \mathbb{C}(x^n, y^n)$ is a division algebra.

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3 is not true for previous two examples.

Commutative Birational Geometry

Algebraic Problem $\longrightarrow$ Solution via Geometry

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Algebraic Problem $\rightarrow$ Solution via Geometry

- **Theorem**[Riemann, Weil]
  Let K be a field of transcendence degree 1 over $\mathbb{C}$.
  Then there exists a unique smooth compact curve C such that $K = \mathbb{C}(C) =$ field of rational functions on C.
  C is a model of K.

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  Then there exists a unique smooth compact curve C such that $K = \mathbb{C}(C) =$ field of rational functions on C.
  C is a model of K.
- **Theorem**[Castelnouvo, Enriques]
  Let K be a field of transcendence degree 2 over $\mathbb{C}$. Then either there exists a unique smooth compact miminal surface S such that $K = \mathbb{C}(S)$,
  or $\mathbb{C}(S) \simeq \mathbb{C}(\mathbb{CP}^1 \times C)$ for a smooth compact curve C.

Noncommutative Birational Geometry

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  Let K be a division algebra such that
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  $\dim_{Z(K)} K < \infty$.
  Then $K = Z(K)$ is commutative.

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- **Theorem**[Daniel Chan,I](Invent.2005)
  Let K be a division algebra such that
  $Z(K)$ is a field of transcendence degree 2 over $\mathbb{C}$
  and $\dim_{Z(K)} K < \infty$. Then
  either there exists a minimal terminal order (vector bundle with algebra structure)
  $\mathcal{A}$ over smooth compact surface Z, such
  that $\mathcal{A} \otimes \mathbb{C}(Z) = K$, unique up to Morita equivalence.
  or K is del Pezzo or Ruled and on list.

Definitions

Order

Let Z be a normal surface over $\mathbb{C}$.

Order $\mathcal{A}$ is a coherent torsion free sheaf on Z
sheaf of $\mathcal{O}_Z$-central algebras.
$\mathcal{A} \otimes \mathbb{C}(Z)$ is a central simple $\mathbb{C}(Z)$-algebra

Maximal Order $\mathcal{A}$ is maximal under inclusion of orders in $\mathcal{A} \otimes \mathbb{C}(Z)$

Discriminant D codimension one locus where $\mathcal{A}$ is not Azumaya.

Ramification Data $R(A) = (\tilde{D} \twoheadrightarrow D \hookrightarrow Z)$
$\mathcal{O}_Z = Z(A)$ centre of $\mathcal{A}$.
D discriminant
$\tilde{D}$ ramified cyclic cover of D
$Gal(\mathbb{C}(\tilde{D}) : \mathbb{C}(D)) = \mathbb{Z}/n\mathbb{Z}$

Terminal Orders

$\mathcal{A}$ is terminal if one of the following equivalent conditions hold

- $discrep(\mathcal{A}) > 0$
- $R(A) = (\tilde{D} \twoheadrightarrow D \hookrightarrow Z)$ satisfies
  - Z is smooth
  - D has normal crossings
  - $\tilde{D}$ only ramifies at the nodes of D with one branch with deg $= e$ and the other has e
- $\mathcal{A}$ is étale locally of the form $\begin{pmatrix} S & \cdots & \cdots & S \\ xS & \ddots & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ xS & \cdots & xS & S \end{pmatrix}$ where

  $S = k_\zeta[x, y]$.

Minimal Orders

$\Delta = \sum \left(1 - \frac{1}{\deg(\tilde{D}_i : D_i)}\right) D_i$
$D = \cup D_i$ irreducible components
$\mathcal{A}$ is minimal if there is no curve $E \subset Z$ such that $E^2 < 0$ and $(K_Z + \Delta).E < 0$.

Applications

- Algebraic Geometry: Birational Classification of
  - Generic $\mathbb{CP}^N$ bundles over surfaces.
  - Deligne-Mumford stacky surfaces with cyclic generic stabilizer.
- Mirror Symmetry: an example
  Calabi-Yau 3fold: $X = Q_0 \cap Q_1 \cap Q_2 \cap Q_3 \subset \mathbb{CP}^7$.
  Mirror: Calabi-Yau Noncommutative 3fold $\mathcal{A}$ with
  $Z(A) \xrightarrow{2} \mathbb{CP}^3$ ramified on V_8.
  $Z(\mathcal{A})$ is singular but $\mathcal{A}$ is smooth.

Fano

Fano $Z(\mathcal{A}) = \mathbb{CP}^2$

$\deg D$	$\deg(\tilde{D} : D)$	Algebra	$\mathrm{BSV}(\mathcal{A})$
3	2	Clifford	$V_{(1,2)} \subset \mathbb{CP}^2 \times \mathbb{CP}^2$
3	≥ 3	Sklyanin	??
4	2	Clifford	$V \xrightarrow{2} \mathbb{CP}^1 \times \mathbb{CP}^2$
4	3	??	(Reid,I)
5	2^+	Clifford	$\mathrm{Bl}_C\, \mathbb{CP}^3$ $\deg C = 7$ $g(C) = 5$
5	2^-	Clifford	$\mathrm{Bl}_{\mathrm{line}}\, V_3, \quad V_3 \subset \mathbb{CP}^3$

Ruled

Ruled $Z(\mathcal{A}) \to C$ with fibres $F \simeq \mathbb{CP}^1$

$D.F$	$\deg(\tilde{D} : D)$	Algebra	$\mathrm{BSV}(\mathcal{A})$
2	2	Clifford	$\mathrm{Bl}_{2\text{ pts}}\, \mathbb{CP}^1 \times \mathbb{CP}^1_{\mathbb{C}(C)}$
2	≥ 3	NC Ruled Surface (VdB)	??
3	2	Clifford	$Bl_{4\text{ pts}} \mathbb{CP}^2_{\mathbb{C}(C)}$

Conjectures

- **Conjecture**[Generalized Iskovskih]
  Let $\mathcal{A}$ be a minimal terminal order, then
  $\mathbb{C}(\mathrm{BSV}(\mathcal{A})) = \mathbb{C}(\mathbb{CP}^N \times C)$ with C a curve or
  $\mathbb{C}(\mathrm{BSV}(\mathcal{A})) = \mathbb{C}(V_3)$ with $V_3 \subset \mathbb{CP}^4$
  $\Leftrightarrow$ $\mathcal{A}$ is ruled or Fano.
- **Conjecture**[Artin]
  K division algebra of transcendence degree two over $\mathbb{C}$. Then
  - $\dim_{Z(K)} K < \infty$ or
  - $K = K(A)$ where A is Sklyanin or quantum ruled.

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