M^{C}KAY GRAPHS

by

DANE FRENETTE

BCS (Theory & Computation) University of New Brunswick

A Thesis Submitted in Partial Fulfilment of
the Requirements for the Degree of

MASTER OF SCIENCE

in the Graduate Academic Unit of Mathematics and Statistics

Supervisor(s):	Dr. Colin Ingalls
Examining Board:	Dr. David Bremner
External Examiner:	Dr. Michael Fleming

This thesis is accepted

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Dean of Graduate Studies

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January 2007

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Abstract

In 1980, John M^{c}Kay described M^{c}Kay graphs and alluded to interesting results that could be obtained through the study of representations of finite groups [McKa]. In this thesis, I characterize M^{c}Kay graphs of degree 1 and present the research done by various people which allows us to understand M^{c}Kay graphs of degree 2. I also discuss that if you view a M^{c}Kay graph as an adjacency matrix, many interesting results can be witnessed. We also use the work performed by [YaYu] to help us catalogue the M^{c}Kay graphs of degree 3.

Introduction

The aim of this work is to study representations of finite groups in order to gain an understanding of M^cKay graphs. The thesis begins by giving background information that is crucial for the understanding of the research performed. Relevant information is then provided on representations of finite groups, $\mathbb{C}G$-modules, and characters of finite groups.

M^cKay graphs are then defined and numerous examples are given from various groups. Through the use of these examples, M^cKay graphs of degree 1 are discussed in detail and we obtain a classification of these. After researching various papers, we gain an understanding of M^cKay graphs of degree 2. From M^cKay [McKa], it is realized that M^cKay graphs of a group $G \subset SL(G,\mathbb{C})$ are one of the extended Dynkin diagrams $\widetilde{A_n}, \widetilde{D_n}, \widetilde{E_6}, \widetilde{E_7}$, or $\widetilde{E_8}$. We then give a generalization of M^cKay's work by Auslander and Reiten [AuRe] to get a description of M^cKay graphs of a group $G \subset GL(G,\mathbb{C})$.

It is shown that we can interpret a M^cKay graph as an adjacency matrix and immediate benefits of this are observed. We show that more complicated M^cKay graphs can be expressed by using operations on simpler M^cKay graphs. We also discover that if we compute the eigenvalues and eigenvectors of the adjacency matrix of a M^cKay graph, one of the eigenvectors corresponds to the dimensions of the irreducible representations and the eigenvalue corresponds to the degree of the M^cKay graph.

We then look at a classification of the finite subgroups of $SL(3,\mathbb{C})$ given by Yau and Yu [YaYu]. All these subgroups are given in this thesis and the M^cKay graphs for these subgroups are described.

It is then proven that if a M^cKay graph is of a faithful representation, it is strongly connected.

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Chapter 1

Representations and Characters of Finite Groups

This chapter provides background information that is essential for the understanding of M^{c}Kay graphs and the research that was performed. It is assumed that the reader is familiar with linear algebra and group theory; however, when such concepts arise, the definitions are often stated for the sake of clarity. I begin by defining a representation of a finite group and giving some examples and important results. $\mathbb{C}G$-modules are then described and it is shown that there is a close relationship between $\mathbb{C}G$-modules and representations. Some details about $\mathbb{C}G$-modules are presented such as irreducibility, Maschke's theorem, and Schur's lemma. I then discuss conjugacy classes as they play an important role in the development of later material. Characters are then analyzed and it is shown that characters are closely linked to representations and $\mathbb{C}G$-modules. The strength of characters is that they often help describe things about representations and $\mathbb{C}G$-modules and their arithmetic is significantly easier. Examples of characters and operations on characters are examined, such as the inner product of characters. We then finish off the chapter by describing character tables and detailing how they present information about the characters in a clear way by taking away redundancy.

The material covered in this chapter was paraphrased from James and Liebeck [JaLi]. If the reader needs further examples or more details, they are encouraged to use this text.

1.1 Representations of Finite Groups

Definition 1.1.1:

Let V be a vector space over $\mathbb{C}$. A representation of a group G is a homomorphism ρ: G $\rightarrow$ GL(V,$\mathbb{C}$). We say that deg(ρ) = dim(V) [JaLi].

GL(V,$\mathbb{C}$) is the general linear group and if dim(V) = n, then GL(V,$\mathbb{C}$) is the set of $n \times n$ invertible matrices with ordinary matrix multiplication. Throughout this paper, we will sometimes write ρ: G $\rightarrow$ GL(n,$\mathbb{C}$) where n = dim(V).

ρ is a homomorphism so we have the following:

1. $\rho(xy) = \rho(x)\rho(y)$ for all $x, y \in$ G.
2. $\rho(1) = I_n$ where 1 is the identity element of G.
3. $\rho(g^{-1}) = (\rho(g))^{-1}$ where $g \in$ G.

Let's look at a few examples of representations.

Example 1.1.2:

Let the trivial representation ρ: G $\rightarrow$ GL(1,$\mathbb{C}$) be defined by $\rho(g) = 1$, for all $g \in$ G.

Example 1.1.3:

If G = S_n, the symmetric group on n variables, then let ρ: G $\rightarrow$ GL(1,$\mathbb{C}$) be defined by

$$\rho(g) = \begin{cases} 1 & \text{if even permutation} \\ -1 & \text{if odd permutation} \end{cases}$$

This is called the alternating representation.

Example 1.1.4:

We can also find matrix representations.

Consider G $= D_8 = \langle a, b | a^4 = b^2 = 1, b^{-1}ab = a^{-1} \rangle$.

Let

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \mathrm{B} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

now,

ρ: G $\rightarrow$ GL(2,$\mathbb{C}$) is given by $\rho : a^i b^j \rightarrow \mathrm{A}^i \mathrm{B}^j$, $(0 \leq i \leq 3, 0 \leq j \leq 1)$.

Notice that $\deg(\rho) = 2$.

Definition 1.1.5:

Let ρ: G $\rightarrow$ GL(n,$\mathbb{C}$) and σ: G $\rightarrow$ GL(m,$\mathbb{C}$). We say that ρ is equivalent to σ if $n = m$ and there exists an invertible $n \times n$ matrix T such that $\sigma(g) = T^{-1}(\rho(g))T$ [JaLi].

The kernel of a representation consists of the group elements g which $\rho(g)$ sends to the identity matrix. We can thus say that $\ker(\rho) = \{g \in \mathrm{G} \; | \rho(g) = I_n\}$.

Definition 1.1.6:

A representation is said to be faithful if $\ker(\rho) = \{1\}$. That is the identity element of the group is the only element for which $\rho(g) = I_n$ [JaLi].

The trivial representation of *Example 1.1.2* illustrates an unfaithful representation. By definition, it sends every group element to the identity. The matrix representation of *Example 1.1.4* shows a faithful representation.

1.2 $\mathbb{C}$G-modules

Definition 1.2.1:

Let V be a vector space over $\mathbb{C}$ and let G be a group. Then V is a $\mathbb{C}$G-module if a multiplication vg where $v \in$ V, $g \in$ G is defined, which satisfies the following conditions:

1. $vg \in$ V
2. $v(gh) = (vg)h$
3. $v * 1 = v$
4. $(\lambda v)g = \lambda(vg)$
5. $(u + v)g = ug + vg$

where $u, v \in$ V; $\lambda \in \mathbb{C}$; $g, h \in$ G [JaLi].

Definition 1.2.2:

Let V be a $\mathbb{C}$G-module and let $\wp$ be a basis of V. For each $g \in$ G, let $[g]_\wp$ denote the matrix of the endomorphism $v \rightarrow vg$ of V, relative to the basis $\wp$ [JaLi].

Theorem 1.2.3:

1. ρ: G $\rightarrow$ GL(n,$\mathbb{C}$) is a representation of G over $\mathbb{C}$ and V = $\mathbb{C}^n$. The vector space V becomes a $\mathbb{C}$G-module if we define $vg = v(\rho(g))$ where $v \in$ V, $g \in$ G.
2. There is a basis $\wp$ of V such that $\rho(g) = [g]_\wp$ for all $g \in$ G.

So, $g \to [g]_\wp$ is a representation of G over $\mathbb{C}$ [JaLi].

Example 1.2.4:

Recall *Example 1.1.4*:

$$\rho(a) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \rho(b) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \mathrm{V} = \mathbb{C}^2$$

Let v_1 and v_2 be the standard basis vectors for $\mathbb{C}^2$.

So, $v_1 a = v_2, v_2 a = -v_1, v_1 b = v_1, v_2 b = -v_2$

Since v_1 and v_2 form a basis $\wp$ over $\mathbb{C}$, $g \to [g]_\wp$ is the representation ρ.

Example 1.2.5:

The trivial $\mathbb{C}$G-module is a 1-dimensional vector space V over $\mathbb{C}$ where $vg = v$, for all $v \in$ V, $g \in$ G.

A $\mathbb{C}$G-module V is faithful if the identity element of G is the only element of g for which $vg = v$, for all $v \in$ V.

Theorem 1.2.6:

If V is a $\mathbb{C}$G-module with bases $\wp$, $\wp'$ and $\rho(g) = [g]_\wp$ and $\sigma(g) = [g]_{\wp'}$, then ρ is equivalent to σ [JaLi].

Definition 1.2.7:

A subset W of a vector space V is a $\mathbb{C}$G-submodule of V if W is a subspace of V and $wg \in$ W, for all $w \in$ W, $g \in$ G [JaLi].

Definition 1.2.8:

A $\mathbb{C}$G-module V is called irreducible if it is non-zero and it has no $\mathbb{C}$G-submodules apart from $\{0\}$ and V. If V has a $\mathbb{C}$G-submodule that is not $\{0\}$ or V, then V is called reducible [JaLi].

We can say that ρ: G $\rightarrow$ GL(n,$\mathbb{C}$) is irreducible if the $\mathbb{C}$G-module, $vg = v(\rho(g))$, is irreducible. If the $\mathbb{C}$G-module is reducible, then ρ is called reducible.

Example 1.2.9:

If $\deg(\rho) = 1$, then ρ is irreducible since the only subrepresentations are $\{0\}$ and V. So, we have seen two representations that have $\deg(\rho) = 1$, namely the trivial and alternating representations (*Examples 1.1.2, 1.1.3*). It follows that these representations are irreducible.

Example 1.2.10:

Let G be a finite group. The vector space $\mathbb{C}$G, with natural multiplication vg where $v \in \mathbb{C}$G, $g \in$ G, is called the regular $\mathbb{C}$G-module. Notice how the group acts on the vector space $\mathbb{C}$G. The representation $\rho : g \to [g]_\wp$ obtained by taking $\wp$ to be the natural basis of $\mathbb{C}$G is called the regular representation of G over $\mathbb{C}$. The regular representation of G is reducible.

<u>*Maschke's Theorem 1.2.11:*</u>

Let G be a finite group and let V be a $\mathbb{C}$G-module. If U is a $\mathbb{C}$G-submodule of V, there is a $\mathbb{C}$G-submodule W such that V = U $\oplus$ W [JaLi].

The consequence of Maschke's theorem that is of particular interest for our cause is that every non-zero $\mathbb{C}$G-module is completely reducible; that is, V $= U_0 \oplus U_1 \oplus \ldots \oplus U_r$ where each U_i is an irreducible $\mathbb{C}$G-module of V.

<u>*Definition 1.2.12:*</u>

$Hom_{\mathbb{C}G}(V, W)$ is the set of all $\mathbb{C}$G-homomorphisms from V to W [JaLi].

<u>*Schur's Lemma 1.2.13:*</u>

Let V and W be irreducible $\mathbb{C}$G-modules.

1. If ϕ: V $\to$ W is a $\mathbb{C}$G-homomorphism, then either ϕ is a $\mathbb{C}$G-isomorphism or $\phi(v) = 0$ for all v $\in$ V.

2. If ϕ: V $\to$ V is a $\mathbb{C}$G-isomorphism, then ϕ is a scalar multiple of the identity 1_V [JaLi].

We then have an immediate consequence of Schur's lemma. If V,W are irreducible $\mathbb{C}G$-modules, then

$$\dim(\mathrm{Hom}_{\mathbb{C}G}(V, W)) = \begin{cases} 1 & \text{if V} \cong \text{W} \\ 0 & \text{if V} \ncong \text{W} \end{cases}$$

1.3 Conjugacy Classes

Definition 1.3.1:

Let $x, y \in$ G. If $y = g^{-1}xg$ for some $g \in$ G, then we say that x is conjugate to y in G [JaLi].

We can denote the set of all elements conjugate to x in G by $x^G = \{g^{-1}xg \mid g \in G\}$. This set is called a conjugacy class.

If we take two group elements then either the conjugacy classes of both elements are equal or they share no elements in common. This means that every group is a union of conjugacy classes where distinct conjugacy classes are disjoint.
That is, G $= x_1^G \cup \ldots \cup x_r^G$, where $x_1^G, \ldots, x_r^G$ are the conjugacy classes and $x_1, \ldots, x_r$ are called class representatives.

Example 1.3.2:

$D_8 = \{1, a, a^2, a^3, b, ab, a^2b, a^3b\}$.
The conjugacy classes for D_8 are: $1^G = \{1\}, a^G = \{a, a^3\}, (a^2)^G = \{a^2\}, b^G = \{b, a^2b\}$, and $ab^G = \{ab, a^3b\}$.
In this example 1, a, a^2, b, and ab are the class representatives for the conjugacy classes.

1.4 Characters

The trace of an $n \times n$ matrix A is given by $tr(A) = \sum_{i=1}^{n} a_{ii}$.

Definition 1.4.1:

Suppose V is a $\mathbb{C}$G-module with basis $\wp$. The character χ: G $\rightarrow \mathbb{C}$ is given by $\chi(g) = tr[g]_{\wp}$ for all $g \in$ G [JaLi].

But the character of V does not depend on the basis because $[g]_{\wp'} = T^{-1}[g]_{\wp}T$. From this we can see that $tr[g]_{\wp'} = tr[g]_{\wp}$ for all $g \in$ G. This tells us that the trace of a matrix is the same regardless of our choice of basis. Using this result, we can define the character of a representation to be $\chi(g) = tr(\rho(g))$.

We say that χ is an irreducible character if it is from an irreducible $\mathbb{C}$G-module. We can also note at this point that conjugate elements of the group have the same character. That is if x is conjugate to y if $y = g^{-1}xg$ for some $g \in$ G, then $\chi(x) = \chi(y)$.

We can now look at a few results about characters:

1. $\chi(1) = \deg(\rho) = \dim(\mathrm{V})$.
2. $\chi(g^{-1}) = \overline{\chi(g)}$.
3. $\chi(g)$ is a real number if g is conjugate to g^{-1}.

4. If g, h are conjugate elements of the group G, then $\chi(g) = \chi(h)$ for all characters χ of G.

5. If χ is a character, then $\overline{\chi}$ is also a character.

6. If χ is an irreducible character, then $\overline{\chi}$ is also an irreducible character [JaLi].

Theorem 1.4.2:

Let ρ: G $\rightarrow$ GL(n,$\mathbb{C}$) be a representation of G and let χ be the character of ρ. Then we have the following statements:

1. For $g \in$ G
   $|\chi(g)| = \chi(1)$ if and only if $\rho(g) = \lambda I_n$ for some $\lambda \in \mathbb{C}$.

2. $\ker(\rho) = \{g \in \text{G} \ |\chi(g) = \chi(1)\}$ [JaLi].

Proposition 1.4.3:

Let ρ and ψ be representations of G. Then $\chi_{\rho\otimes\psi} = \chi_\rho\chi_\psi$ and $\chi_{\rho\oplus\psi} = \chi_\rho + \chi_\psi$ [JaLi].

The proposition above tells us that if we are trying to multiply or add representations, we must use the tensor product or direct sum since we are either multiplying or adding vector spaces. If we are trying to multiply or add characters though, we can use regular multiplication or addition to accomplish this feat. We can now define another operation on characters, the inner product or pairing of characters.

Definition 1.4.4:

Let χ and ψ be characters of G. The inner product of χ and ψ is given by

$$\langle \chi, \psi \rangle = \tfrac{1}{|G|} \sum_{g \in G} \chi(g)\overline{\psi(g)}$$

The inner product also satisfies some conditions:

1. $\langle \chi, \psi \rangle = \overline{\langle \psi, \chi \rangle}$
2. $\langle \lambda_1 \chi + \lambda_2 \psi, \phi \rangle = \lambda_1 \langle \chi, \phi \rangle + \lambda_2 \langle \psi, \phi \rangle$
3. $\langle \chi, \chi \rangle > 0$ iff $\chi \neq 0$.

The following theorem illustrates how any character can be composed by adding other irreducible characters together.

Theorem 1.4.5:

Let $\chi_1, \ldots, \chi_r$ be the irreducible characters of G. If ψ is any character of G, then $\psi = d_1\chi_1 + \ldots + d_r\chi_r$ where the integers $d_i = \langle \psi, \chi_i \rangle$ for $1 \leq i \leq r$ [JaLi].

Theorem 1.4.6:

Let V be a $\mathbb{C}$G-module with character χ. Then V is irreducible if and only if $\langle \chi, \chi \rangle$ = 1 [JaLi].

Theorem 1.4.7:

Suppose that V and W are $\mathbb{C}$G-modules with characters χ and ψ respectively. Then V $\cong$ W if and only if $\chi = \psi$ [JaLi].

Theorem 1.4.8:

Let V and W be $\mathbb{C}$G-modules with characters χ and ψ respectively. Then $\dim(Hom_{\mathbb{C}G}(V, W)) = \langle \chi, \psi \rangle$ [JaLi].

We are now in a position to comment on the number of irreducible characters of a group G. The number of irreducible characters of G is equal to the number of conjugacy classes of G.

Definition 1.4.9:

Suppose $\chi_1, \ldots, \chi_r$ are all the irreducible characters of G. The regular character $\chi_{reg} = d_1\chi_1 + \ldots + d_r\chi_r$ where χ_i is the irreducible $\mathbb{C}$G-module V_i and $d_i = \chi_i(1)$. Notice that $\chi_{reg}(1) = |\text{G}|$ and that χ_{reg} is a reducible character [JaLi].

1.5 Character Tables

Character tables allow us to gather all of the information about characters of a group in a clear, concise format that eliminates a lot of redundancy by utilizing the conjugacy classes. The columns of the table correspond to the r conjugacy classes of G. The rows describe the characters of the distinct irreducible representations $\chi_1, \ldots, \chi_r$ of G. It is known that the number of irreducible characters of G is equal to the number of conjugacy classes of G, so we can be guaranteed that the table is a $r \times r$ square table. By convention, the first representation listed in the character table is the trivial representation and the first column always corresponds to the conjugacy class 1^G. We also say that the first column of the character table contains the degrees of the irreducible representations.

Definition 1.5.1:

Let $\chi_1, \ldots, \chi_r$ be the irreducible characters of G and let $g_1, \ldots, g_r$ be the class representatives of the conjugacy classes of G. The $r \times r$ matrix whose ij^{th} entry is $\chi_i(g_j)$ is called the character table of G [JaLi].

Example 1.5.2:

The complete character table for D_8 is given below:

g	1	a^2	a	b	ab
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	1	-1	1	-1
χ_4	1	1	-1	-1	1
χ_5	2	-2	0	0	0

Chapter 2

M^{c}Kay Graphs

This chapter describes what is already known about M^{c}Kay graphs and the research that I did while working towards a Master's degree. In the first section, M^{c}Kay graphs are defined. Numerous M^{c}Kay graphs are then presented from a variety of groups. In the second section, the M^{c}Kay graphs of degree 1 are characterized. The third section discusses results from three papers by M^{c}Kay [McKa], Auslander and Reiten [AuRe], and Reiten and Van den Bergh [ReVB]. These three papers allow us to understand M^{c}Kay graphs of degree 2. In the following section, we consider the encoding of a M^{c}Kay graph in its adjacency matrix, and use this to notice and prove some interesting properties about these graphs. The next section comments on the classification of finite subgroups of SL(3,$\mathbb{C}$) that was studied in Yau and Yu [YaYu]. Their findings are presented in this thesis and are used to help catalogue the M^{c}Kay graphs of degree 3. The chapter then concludes by discussing some conditions that affect the shape of a M^{c}Kay graph.

2.1 M^{c}Kay Graphs

We begin this chapter by defining a M^{c}Kay graph.

Definition 2.1.1:

Suppose $\rho_1, \ldots, \rho_r$ is the set of non-isomorphic irreducible representations of G. For any representation τ of G over an algebraically closed field k, we denote by $\text{mult}_i(\tau)$ the dimension of $Hom_{\mathbb{C}G}(\rho_i, \tau)$ as a $\mathbb{C}$-vector space. The M^{c}Kay graph McK(G,τ) is defined to be an oriented graph whose vertices are $\rho_i, (1 \leq i \leq r)$, and there are μ arrows from ρ_i to ρ_j when $\mu = \text{mult}_j(\tau \otimes \rho_i)$ [Yosh].

Recall that $Hom_{\mathbb{C}G}(\rho_i, \tau)$ is the set of homomorphisms from the representation ρ_i to the representation τ. Since we're considering the set of non-isomorphic irreducible representations, the M^{c}Kay graph describes the multiplicity between these representations by connecting the corresponding vertices with edges.

The degree of a M^{c}Kay graph McK(G,τ) is, by definition, deg(τ). Note that this definition of degree is not the same as the usual notion of degree of a vertex of a graph. It is also seen that a representation τ is chosen when describing M^{c}K(G,τ) and for the remainder of the thesis we will call this representation our favorite representation.

When drawing M^{c}Kay graphs I will use the following conventions:

- Arrowheads will not be drawn if $\langle \chi_\tau \chi_{\rho_i}, \chi_{\rho_j} \rangle = \langle \chi_\tau \chi_{\rho_j}, \chi_{\rho_i} \rangle$. A line connecting the two vertices will be drawn.

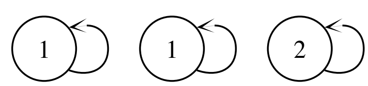
- The degree of the representation will be marked inside the vertex [Noor].

Example 2.1.2:

Consider the character table for D_6, the dihedral group with 6 elements.

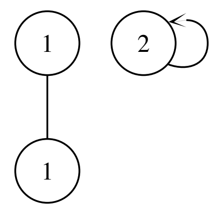
g	1	a	b
χ_1	1	1	1
χ_2	1	1	-1
χ_3	2	-1	0

If our favorite representation is χ_1, then McK(D_6, χ_1) is:



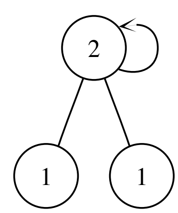
Notice that the degree of McK(D_6, χ_1) is one.

If our favorite representation is χ_2, then McK(D_6, χ_2) is:



This M^{c}Kay graph is also of degree one.

And finally, if our favorite representation is χ_3, then McK(D_6, χ_3) is:



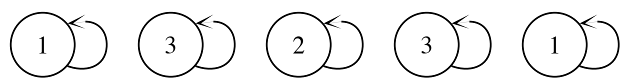
This M^{c}Kay graph is of degree two.

<u>*Example 2.1.3:*</u>

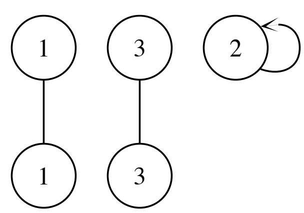
Consider the character table for S_4, the symmetric group on 4 variables.

g	1	(12)	(123)	(12)(34)	(1234)
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	2	0	-1	2	0
χ_4	3	1	0	-1	-1
χ_5	3	-1	0	-1	1

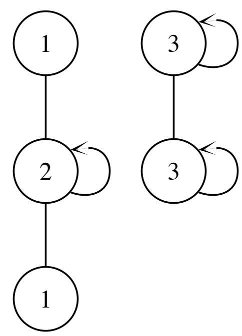
McK(S_4, χ_1):



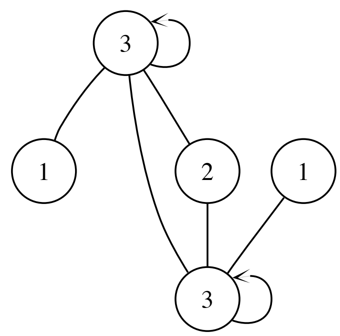
McK(S_4, χ_2):



$\mathrm{McK}(S_4, \chi_3)$:



$\mathrm{McK}(S_4, \chi_4)$ and $\mathrm{McK}(S_4, \chi_5)$ are:



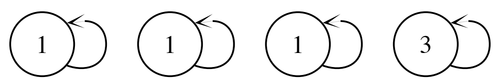
<u>*Example 2.1.4:*</u>

Consider the character table for A_4, the alternating group on 4 variables.

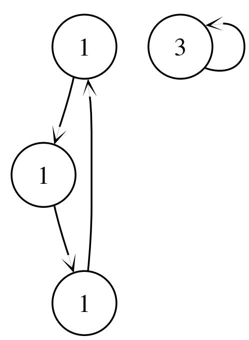
g	1	(12)(34)	(123)	(132)
χ_1	1	1	1	1
χ_2	1	1	ω	ω^2
χ_3	1	1	ω^2	ω
χ_4	3	-1	0	0

$$\omega = e^{2\pi i/3}$$

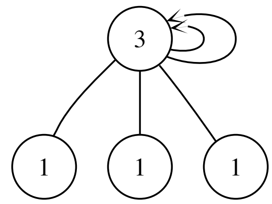
McK(A_4, χ_1):



McK(A_4, χ_2) and McK(A_4, χ_3) are:



McK(A_4, χ_4):



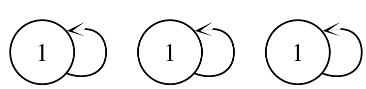
Example 2.1.5:

Consider the character table of C_3, the cyclic group of order 3.

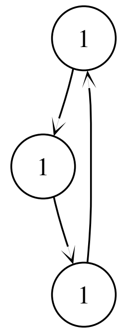
g	1	a	a^2
χ_1	1	1	1
χ_2	1	ω	ω^2
χ_3	1	ω^2	ω

$$\omega = e^{2\pi i/3}$$

McK(C_3, χ_1):



McK(C_3, χ_2) and McK(C_3, χ_3) are:



2.2 M^{c}Kay Graphs of Degree 1

After seeing numerous examples of M^{c}Kay graphs, we are now in a position to comment on the M^{c}Kay graphs of degree 1.

A recurring M^{c}Kay graph that has appeared in all four groups has been the trivial M^{c}Kay graph. The trivial M^{c}Kay graph corresponds to the graphs that just have vertices with self-loops. Since the number of vertices in a M^{c}Kay graph is equal to the number of irreducible representations of G, the trivial McK(G,τ) is produced when our favorite representation τ is the trivial representation.

Claim 2.2.1:

For any group G, there is an irreducible representation τ such that McK(G,τ) is the trivial graph.

Proof:

For every group a character table can be produced. The trivial representation is always an irreducible representation for a group. If we choose the trivial representation ρ_1 as our favorite representation the result follows since $\text{mult}_j(\rho_1 \otimes \rho_i) = \text{mult}_j(\rho_i) = 1$ (if i = j) or 0 (if i $\neq$ j). $\square$

The trivial representation is not necessarily the only representation of degree 1. At this point, it is worthwhile mentioning that representations of degree one are called linear representations.

Theorem 2.2.2:

If τ is a linear representation, then McK(G,τ) is a disjoint union of directed cycles where all the vertices in a cycle correspond to representations of the same degree.

Before we can prove this theorem, we need to introduce the following proposition.

Proposition 2.2.3:

Let χ be a character and λ be a linear character. Then the product $\chi\lambda$ is a character of G. Furthermore, if χ is irreducible, then so is $\chi\lambda$ [JaLi].

Proof of Proposition 2.2.3:

Let ρ: G $\rightarrow$ GL(n,$\mathbb{C}$) be a representation with character χ. Define $\rho\lambda$: G $\rightarrow$ GL(n,$\mathbb{C}$) by $\rho\lambda(g) = \lambda(g)\rho(g)$ where $g \in$ G.

We can see that $\rho\lambda(g)$ is the matrix $\rho(g)$ multiplied by the complex number $\lambda(g)$. Since ρ and λ are homomorphisms, it follows that $\rho\lambda$ is a homomorphism. The matrix $\lambda(g)\rho(g)$ has a trace $\lambda(g)tr(\rho(g))$, which is $\lambda(g)\chi(g)$. We can now say that $\rho\lambda$ is a representation of G with character $\chi\lambda$.

We know that for all $g \in$ G, the complex number $\lambda(g)$ is a root of unity, so $\lambda(g)\overline{\lambda(g)}$ $= 1$.

$$\langle \chi\lambda, \chi\lambda \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g)\lambda(g)\overline{\chi(g)\lambda(g)} = \frac{1}{|G|} \sum_{g \in G} \chi(g)\overline{\chi(g)} = \langle \chi, \chi \rangle$$

By *Theorem 1.4.6*, ρ is irreducible iff $\langle \chi, \chi \rangle = 1$. It follows that $\chi\lambda$ is irreducible if χ is irreducible. □

Proof of Theorem 2.2.2:

Let G be a group, τ be our favorite representation, and $\rho_1, \ldots, \rho_r$ be the irreducible representations of G. In order to prove this statement, we first have to show that the number of arrows entering and exiting any vertex is one.

Suppose τ is any linear representation. Then $\text{mult}_i(\tau \otimes \rho_j) = \text{mult}_i(\widehat{\rho_j})$, where $\tau \otimes \rho_j = \widehat{\rho_j}$. Since τ is a linear representation and ρ_j is an irreducible representation of G, *Proposition 2.2.3* tells us that $\widehat{\rho_j}$ is also an irreducible representation of G. It is clear that $\dim(\widehat{\rho_j}) = \dim(\rho_j)$. If ρ_j is an irreducible representation of dimension n, then $\rho_j \otimes \tau$ is another irreducible representation of dimension n. So there is an arrow from ρ_j to $\widehat{\rho_j}$. But only one arrow exits every representation ρ_i because there are only a finite number of irreducible representations for G.

We want to show that there is only one arrow entering every ρ_i. Well, $\text{mult}_j(\tau \otimes \overline{\rho_i}) = \text{mult}_j(\widetilde{\rho_i})$, where $\tau \otimes \overline{\rho_i} = \widetilde{\rho_i}$. *Proposition 2.2.3* says that $\widetilde{\rho_i}$ is an irreducible representation of G. Taking the tensor product of the conjugate of ρ_i, for all $\rho_i \in \{\rho_1, \ldots, \rho_r\}$, reverses the direction of the arrows of the M^{c}Kay graph. Using this M^{c}Kay graph, we can say that every representation ρ_i has only one arrow exiting it. By performing the tensor product by the conjugate again, we retrieve the original M^{c}Kay. Since each representation ρ_i had one arrow exiting it in the "reversed" M^{c}Kay graph, each representation ρ_i will have only one arrow entering it in the original M^{c}Kay graph.

We know that the number of arrows entering and exiting any vertex is one. We also know that if $\dim(\rho_i) = \dim(\rho_j)$, then the two vertices ρ_i and ρ_j are connected by an edge. This happens because τ is a linear representation and it was shown that the tensor product calculation, $\tau \otimes \rho_y$, obtained a representation that had the same dimension as ρ_y. So by [West, 1.4.5], we have that the M^{c}Kay graphs are composed of directed cycles of the same dimension. $\square$

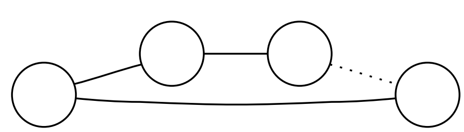
2.3 M^{c}Kay Graphs of Degree 2

Now that we've commented on the M^{c}Kay graphs of degree one, we can discuss what is already known about M^{c}Kay graphs of degree two.

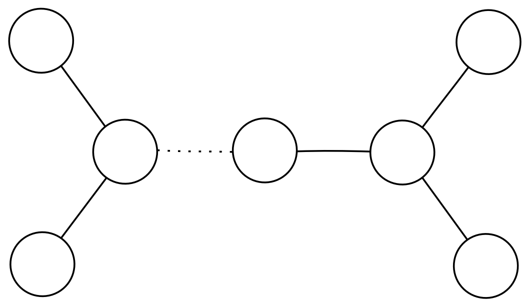
In 1980, John M^{c}Kay wrote a paper [McKa] in which he speculated that through the study of representation theory, the understanding of finite groups would greatly improve. M^{c}Kay thought that this would help with the problem of classifying finite groups and lead to new proofs that would be shorter than their counterparts of the day. M^{c}Kay observed that if a finite group G $\subset$ SL(2,$\mathbb{C}$), then the resulting M^{c}Kay graph is one of the extended Dynkin diagrams $\widetilde{A_n}, \widetilde{D_n}, \widetilde{E_6}, \widetilde{E_7}$, or $\widetilde{E_8}$. The reader should note that the special linear group, SL(n,$\mathbb{C}$), is a subgroup of GL(n,$\mathbb{C}$) consisting of matrices with determinant 1.

Each extended Dynkin diagram has n+1 vertices. The following are the extended Dynkin diagrams listed above [Yosh]:

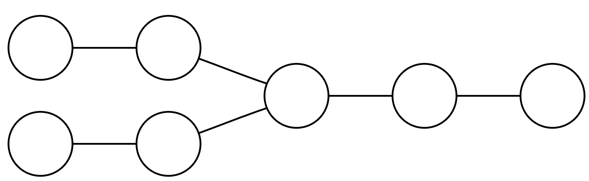
$\widetilde{A_n}$ (cyclic):



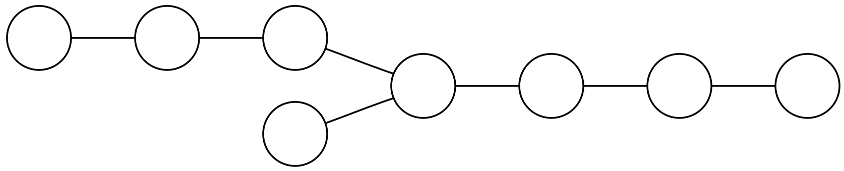
$\widetilde{D_n}$ (binary dihedral):



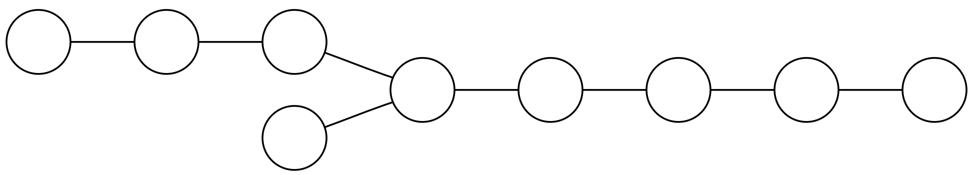
$\widetilde{E_6}$ (binary tetrahedral):



$\widetilde{E_7}$ (binary octahedral):



$\widetilde{E_8}$ (binary icosahedral):



In 1986, Auslander and Reiten [AuRe] extended M^{c}Kay's observations to arbitrary two dimensional representations. Let k be an algebraically closed field, G a finite group, and ψ: G $\rightarrow$ GL(m,k) be our favorite representation. We know that McK(G, ψ) has vertices $\rho_1, \ldots, \rho_r$ which correspond to the irreducible representations of G. These irreducible representations of G have $\chi_1, \ldots, \chi_r$ as characters. The separated M^{c}Kay graph, $\overline{McK(G, \psi)}$, has vertices $\rho_1, \ldots, \rho_r$, $\rho'_1, \ldots, \rho'_r$ and for each arrow from ρ_i to ρ_j in McK(G, ψ), there is an arrow from ρ_i to ρ'_j in $\overline{McK(G, \psi)}$.

Definition 2.3.1:

The underlying graph of a directed graph D is the graph obtained by ignoring the orientation of D [West].

Auslander and Reiten obtained the main result of their paper by studying separated M^{c}Kay graphs and their underlying graphs. Their main result is presented below:

<u>*Theorem 2.3.2:*</u>

Given ψ: G $\to$ GL(m,k) a representations of G and k an algebraically closed field then,

- if $m = 2$, the underlying graph of the separated M^{c}Kay graph $\overline{McK(G,\psi)}$ is a finite union of copies of the extended Dynkin diagrams $\widetilde{A_n}$, $\widetilde{D_n}$, $\widetilde{E_6}$, $\widetilde{E_7}$, and $\widetilde{E_8}$.

- if $m > 2$, $\overline{McK(G,\psi)}$ is a disjoint union of graphs which are not Dynkin or extended Dynkin diagrams.

In order to gain a better understanding about M^{c}Kay graphs of degree 2, one could examine [Coxe]. In this book, Coxeter classifies the finite subgroups of GL(2,$\mathbb{C}$) and paves the way for the classification of McK(G,$\mathbb{C}$), where G $\subset$ GL(2,$\mathbb{C}$).

Another paper which discusses M^{c}Kay graphs of degree 2 is [ReVB]. The material in their paper is briefly summarized in this thesis. The reader should refer to [ReVB] for a more indepth analysis.

Reiten and Van den Bergh wrote a paper where they classified algebras of tame orders of dimension 2. In this paper, it was proved that if an algebra is a tame order of finite representation type, then the AR quiver of the algebra must be of the form ZΔ/G. In this form, Δ is the extended Dynkin diagram $\widetilde{D_n}$, $\widetilde{E_6}$, $\widetilde{E_7}$, $\widetilde{E_8}$, or A_∞^∞ and G is an automorphism group of the translation quiver. It is also said that these translation quivers are equivariant AR quivers. It is described in [Yosh] that if $G \subset GL(2,\mathbb{C})$, then these equivariant translation quivers are the M^{c}Kay graphs of degree 2.

2.4 M^cKay Graphs as Adjacency Matrices

Up to this point we've seen many M^cKay graphs and observed properties related to the degree of the M^cKay graph. Now we can examine what happens to the M^cKay graphs when certain operations are performed.

Definition 2.4.1:

Let G be a group, ψ be our favorite representation, and $\rho_1 \ldots, \rho_r$ be the irreducible representations with characters $\chi_1, \ldots, \chi_r$. The $r \times r$ adjacency matrix of a directed graph is a matrix with rows and columns labeled by graph vertices. The ij^{th} entry of the matrix will contain the number of edges from ρ_i to ρ_j and this is computed using $a_{ij} = \langle \psi\chi_i, \chi_j \rangle$ [West].

Throughout this thesis, we will denote the adjacency matrix of McK(G,χ) by Adj(G,χ). It should be noted that every directed graph can be represented as an adjacency matrix. From the above definition, we see that entries in the adjacency matrix are the multiplicities between the representations of the M^cKay graph.

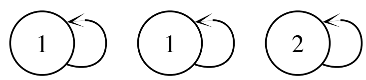
Example 2.4.2:

Let $\rho_1, \ldots, \rho_r$ be all the irreducible representations of a group G. We have already seen that a trivial M^cKay graph exists for each group G. Thus, the M^cKay graph consists of r vertices, each vertex having only a self-loop. We then produce a $r \times r$ adjacency matrix with ones down the diagonal.

Another interesting observation about adjacency matrices is that if there are no self-loops are in the graph, then the adjacency matrix will contain zeros on the diagonal.

Example 2.4.3:

Recall McK(D_6, χ_1):



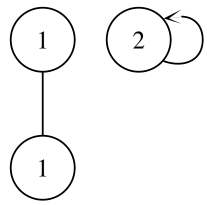
so,

$$\mathrm{Adj}(\mathrm{D}_6, \chi_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Notice that the trivial M^{c}Kay graph corresponds to the identity matrix.

Example 2.4.4:

Recall McK(D_6, χ_2):

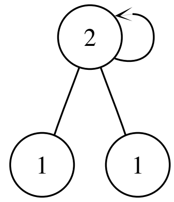


so,

$$\text{Adj}(\text{D}_6, \chi_2) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Example 2.4.5:

Recall McK(D_6, χ_3):

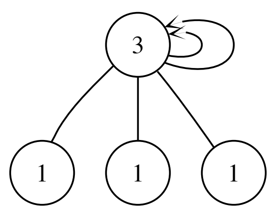


so,

$$\mathrm{Adj}(\mathrm{D}_6, \chi_3) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Example 2.4.6:

Recall $\mathrm{McK}(A_4, \chi_4)$:

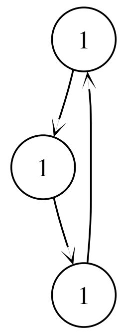


so,

$$\mathrm{Adj}(\mathrm{A}_4, \chi_4) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}$$

Example 2.4.7:

Recall McK(C_3, χ_2):



so,

$$\mathrm{Adj}(\mathrm{C}_3, \chi_2) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Notice that there are no self-loops in McK(C_3, χ_2) and there are zeros on the diagonal of Adj(C_3, χ_2).

Theorem 2.4.8:

Let G be a group, ψ and ϕ be characters (not necessarily irreducible) of G, and $\chi_1, \ldots, \chi_r$ be the irreducible characters of G. Given two M^{c}Kay graphs, McK(G,ψ) and McK(G,ϕ), then McK(G,$\psi + \phi$) has adjacency matrix Adj(G,ψ) + Adj(G,ϕ).

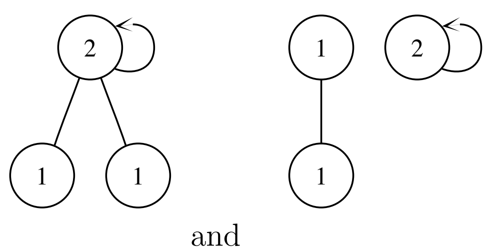
Proof:

We know that every M^{c}Kay graph can be represented as an adjacency matrix. So for McK(G,ψ) and McK(G,ϕ) we have Adj(G,ψ) and Adj(G,ϕ) respectively.
We know that the addition of matrices is done by adding the components of each matrix. If we wanted to calculate the ij^{th} component of Adj(G,ψ), it would be done by $\langle \psi\chi_i, \chi_j \rangle$. Similarily, the ij^{th} component of Adj(G,ϕ) would be found by $\langle \phi\chi_i, \chi_j \rangle$.

So, $\langle \psi\chi_i, \chi_j \rangle + \langle \phi\chi_i, \chi_j \rangle = \langle \psi\chi_i + \phi\chi_i, \chi_j \rangle = \langle (\psi + \phi)\chi_i, \chi_j \rangle$ computes the ij^{th} entry of Adj(G,$\psi + \phi$).
Our result follows since we know that Adj(G,$\psi + \phi$) is the adjacency matrix for McK(G,$\psi + \phi$). □

Example 2.4.9:

Recall McK(D_6, χ_3) and McK(D_6, χ_2):

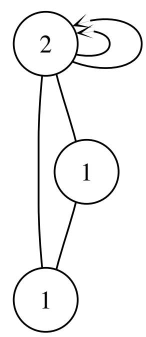


We saw in *Examples 2.4.4* and *2.4.5* that these M^{c}Kay graphs have the following adjacency matrices.

$$\mathrm{Adj}(\mathrm{D}_6, \chi_3) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \mathrm{Adj}(D_6, \chi_2) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\text{thus, } \mathrm{Adj}(D_6, \chi_3) + \mathrm{Adj}(D_6, \chi_2) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

so, $\mathrm{McK}(D_6, \chi_3 + \chi_2)$ is



<u>*Theorem 2.4.10:*</u>

Let G be a group, ψ and ϕ be characters (not necessarily irreducible) of G, and $\chi_1, \ldots, \chi_r$ be the irreducible characters of G. Given two $\mathrm{M^cKay}$ graphs, McK(G,ψ) and McK(G,ϕ), then McK(G,$\psi\phi$) has adjacency matrix Adj(G,ψ)$\times$Adj(G,ϕ).

In order to prove the theorem, we need the following lemma.

Lemma 2.4.11:

Let G be a group, ψ and ϕ be two characters of G, and $\chi_1 \dots \chi_r$ be the irreducible characters of G. Then $\langle \psi, \phi \rangle = \sum_{k=1}^{r} \langle \psi, \chi_k \rangle \langle \chi_k, \phi \rangle$.

Proof of Theorem 2.4.10:

Let Adj(G,ψ) and Adj(G,ϕ) be the adjacency matrices for McK(G,ψ) and McK(G,ϕ) respectively.

We know that computing the ij^{th} entry of Adj(G,ψ)$\times$Adj(G,ϕ) is done by performing a dot product on the i^{th} row of Adj(G,ψ) and the j^{th} column of Adj(G,ϕ).

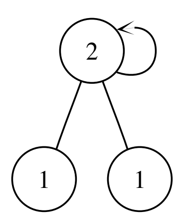
By using *Lemma 2.4.11*, $\sum_{k=1}^{r} \langle \psi\chi_i, \chi_k \rangle \langle \phi\chi_k, \chi_j \rangle = \sum_{k=1}^{r} \langle \psi\chi_i, \chi_k \rangle \langle \chi_k, \overline{\phi}\chi_j \rangle = \langle \psi\chi_i, \overline{\phi}\chi_j \rangle$ $= \langle \phi\psi\chi_i, \chi_j \rangle = \langle \psi\phi\chi_i, \chi_j \rangle$ computes the ij^{th} entry of Adj(G,$\psi\phi$).

Our result follows since we know that Adj(G,$\psi\phi$) is the adjacency matrix for McK(G,$\psi\phi$).

$\square$.

Example 2.4.12:

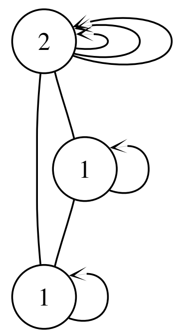
Recall McK(D_6, χ_3):



$$\text{and Adj}(D_6, \chi_3) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$\mathrm{Adj}(D_6,\chi_3)\times\ \mathrm{Adj}(D_6,\chi_3) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \times \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 3 \end{pmatrix}$$

so, McK($D_6,\chi_3\chi_3$) is:



These last two results tell us how we can contruct new M^{c}Kay graphs from existing M^{c}Kay graphs. It also gives us a way to deconstruct more complicated M^{c}Kay graphs into operations involving simpler M^{c}Kay graphs.

Theorem 2.4.13:

Let G be a group, ψ be an irreducible representation with character τ and let $\rho_1,\ldots,\rho_r$ be the irreducible representations of G with characters $\chi_1,\ldots,\chi_r$. Let Adj(G,τ) be the $r\times r$ adjacency matrix of McK(G,τ). If we compute the eigenvalues and eigenvectors of the adjacency matrix, we will find that one of the eigenvectors consists of the dimensions of the irreducible representations and the corresponding eigenvalue is $\tau(1)$

[McKa].

Proof:

We want to show that

$$\begin{pmatrix} a_{11} & \cdots & a_{1r} \\ \vdots & \ddots & \vdots \\ a_{r1} & \cdots & a_{rr} \end{pmatrix} \begin{pmatrix} \chi_1(1) \\ \vdots \\ \chi_r(1) \end{pmatrix} = \tau(1) \begin{pmatrix} \chi_1(1) \\ \vdots \\ \chi_r(1) \end{pmatrix}$$

Let's take a look at the LHS of the equation,

$$\begin{pmatrix} a_{11} & \cdots & a_{1r} \\ \vdots & \ddots & \vdots \\ a_{r1} & \cdots & a_{rr} \end{pmatrix} \begin{pmatrix} \chi_1(1) \\ \vdots \\ \chi_r(1) \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{r} a_{1j}\chi_j(1) \\ \vdots \\ \sum_{j=1}^{r} a_{rj}\chi_j(1) \end{pmatrix}$$

We know that $a_{ij} = \langle \tau\chi_i, \chi_j \rangle = mult_j(\tau\chi_i)$. We also know that $\chi_j(1) = dim(V_j)$.

So,

$$= \begin{pmatrix} \sum_{j=1}^{r} mult_j(\tau\chi_1)dim(\rho_j) \\ \vdots \\ \sum_{j=1}^{r} mult_j(\tau\chi_r)dim(\rho_j) \end{pmatrix}$$

Let's take a look at the RHS of the equation,

$$\tau(1)\begin{pmatrix} \chi_1(1) \\ \vdots \\ \chi_r(1) \end{pmatrix} = dim(\psi)\begin{pmatrix} dim(\rho_1) \\ \vdots \\ dim(\rho_r) \end{pmatrix} = \begin{pmatrix} dim(\psi \otimes \rho_1) \\ \vdots \\ dim(\psi \otimes \rho_r) \end{pmatrix}$$

$$= \begin{pmatrix} dim(\rho_1^{mult_1(\psi\otimes\rho_1)} \oplus \cdots \oplus \rho_r^{mult_r(\psi\otimes\rho_1)}) \\ \vdots \\ dim(\rho_1^{mult_1(\psi\otimes\rho_r)} \oplus \cdots \oplus \rho_r^{mult_r(\psi\otimes\rho_r)}) \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{j=1}^{r} mult_j(\tau\chi_1)dim(\rho_j) \\ \vdots \\ \sum_{j=1}^{r} mult_j(\tau\chi_r)dim(\rho_j) \end{pmatrix}$$

$\therefore LHS = RHS$

□

Definition 2.4.14:

Let v be a vertex of a directed graph. The indegree of v is the number of edges entering v [West].

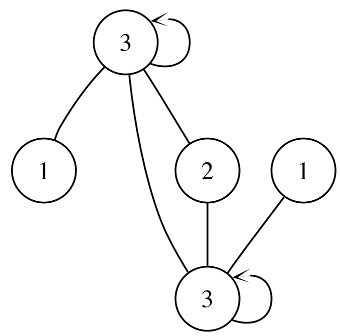
A consequence *Theorem 2.4.13* is stated in the corollary below.

Corollary 2.4.15:

Let G be a group, ψ be an irreducible representation with character τ and let $\rho_1, \ldots, \rho_r$ be the irreducible representations of G with characters $\chi_1, \ldots, \chi_r$. We have that the indegree of vertex ρ_j is equal to the product of the degree of character ρ_j and the degree of M^{c}K(G, ψ). This can also be described as $\sum_{i=1}^{r} \text{mult}_j(\tau\chi_i) = \deg(\rho_j)\times\deg(\psi)$.

Example 2.4.16:

Consider McK(S_4, χ_4):



The first thing to observe is that the degree of this M^{c}Kay graph is 3. Each vertex representing a 1-dimensional irreducible representation is connected to a 3-dimensional vertex, so they have an indegree of 3.

The 2-dimensional vertex has both 3-dimensional vertices connected to it, so there is an indegree of 6 at this vertex and 6 is a multiple of 3.

Both 3-dimensional vertices have the same connections. There is a self-loop, an edge from the other 3-dimensional vertex, an edge from a 2-dimensional vertex, and an edge from a 1-dimensional vertex. There is an indegree of 9 at this vertex and 9 is a multiple of 3.

Theorem 2.4.17:

Let G be a group, ψ be an irreducible representation with character τ and let $\rho_1, \ldots, \rho_r$ be the irreducible representations of G with characters $\chi_1, \ldots, \chi_r$. Let Adj(G,τ) be the $r \times r$ adjacency matrix of McK(G,τ). If we compute the eigenvalues and eigenvectors of the adjacency matrix, we will find that the columns of the character table of G are eigenvectors and the corresponding eigenvalues are $\tau(g)$.

<u>*Proof:*</u>

We want to show that

$$\begin{pmatrix} a_{11} & \cdots & a_{1r} \\ \vdots & \ddots & \vdots \\ a_{r1} & \cdots & a_{rr} \end{pmatrix} \begin{pmatrix} \chi_1(g) \\ \vdots \\ \chi_r(g) \end{pmatrix} = \tau(g) \begin{pmatrix} \chi_1(g) \\ \vdots \\ \chi_r(g) \end{pmatrix}$$

Let's take a look at the LHS of the equation,

$$\begin{pmatrix} a_{11} & \cdots & a_{1r} \\ \vdots & \ddots & \vdots \\ a_{r1} & \cdots & a_{rr} \end{pmatrix} \begin{pmatrix} \chi_1(g) \\ \vdots \\ \chi_r(g) \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{r} a_{1j}\chi_j(g) \\ \vdots \\ \sum_{j=1}^{r} a_{rj}\chi_j(g) \end{pmatrix}$$

We know that $a_{ij} = \langle \tau\chi_i, \chi_j \rangle$.

So,

$$= \begin{pmatrix} \sum_{j=1}^{r} \langle \tau\chi_1, \chi_j \rangle \chi_j(g) \\ \vdots \\ \sum_{j=1}^{r} \langle \tau\chi_r, \chi_j \rangle \chi_j(g) \end{pmatrix}$$

But we if we have any character τ and irreducible characters $\chi_1, \ldots, \chi_r$, it can be expressed as $\tau = \sum_{i=1}^{r} \langle \tau, \chi_i \rangle \chi_i$ (*Theorem 1.4.5*).

So,

$$= \begin{pmatrix} \tau(g)\chi_1(g) \\ \vdots \\ \tau(g)\chi_r(g) \end{pmatrix}$$

Let's take a look at the RHS of the equation,

$$\tau(g) \begin{pmatrix} \chi_1(g) \\ \vdots \\ \chi_r(g) \end{pmatrix} = \begin{pmatrix} \tau(g)\chi_1(g) \\ \vdots \\ \tau(g)\chi_r(g) \end{pmatrix}$$

$\therefore LHS = RHS$

□

I also worked on trying to develop an algorithm that would calculate the columns of the character table if we are given the M^{c}Kay graph. I was only able to describe an algorithm that partially achieved this result. The following two examples describe the procedure and outline its shortcomings.

Example 2.4.18:

Recall Adj(D$_6$, χ_3):

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

The eigenvalues of this matrix are λ_1 = -1, λ_2 = 0, λ_3 = 2 and the eigenvectors are v_{λ_1} = [-1, -1, 1]^T, v_{λ_2} = [-1, 1, 0]^T, and v_{λ_3} = [1/2, 1/2, 1]^T. Up to scaling and reordering, these eigenvectors correspond to the columns of the character table of D$_6$.

g	1	a	b
χ_1	1	1	1
χ_2	1	1	-1
χ_3	2	-1	0

By observing the character table for D_6 above, we can see that if we scale v_{λ_1} by -1 and v_{λ_3} by 2 and then put the eigenvectors in a matrix, we obtain the irreducible characters of D_6.

Through experimentation, it was noticed that the irreducible characters of a group could be retrieved from a M^{c}Kay graph if the eigenvalues were distinct. The next example demonstrates the algorithm's inability to cope with repeated eigenvalues.

Example 2.4.19:

Recall Adj(A_4, χ_4);

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}$$

The eigenvalues of this matrix are λ_1 = -1, $\lambda_2 = \lambda_3 = 0$, and $\lambda_4 = 3$ and the eigenvectors are v_{λ_1} = [-1, -1, -1, 1]^T, $v_{\lambda_2} = v_{\lambda_3}$ = [-1, 0, 1, 0]^T, and v_{λ_4} = [1/3, 1/3, 1/3, 1]^T. Up to scaling and reordering, the eigenvectors only reveal certain columns of the character table of A_4.

g	1	(12)(34)	(123)	(132)
χ_1	1	1	1	1
χ_2	1	1	ω	ω^2
χ_3	1	1	ω^2	ω
χ_4	3	-1	0	0

$$\omega = e^{2\pi i/3}$$

By observing the character table for A_4 above, it can be seen that if we scale v_{λ_1} by -1 and v_{λ_4} by 3, we obtain (up to reordering) the column vectors of the character table corresponding to the conjugacy classes 1 and (12)(34). However, no scaling can be done to gain the column vectors associated with the conjugacy classes (123) and (132). The repeated eigenvectors do not help in calculating the column vectors of the character table.

2.5 M^{c}Kay Graphs of Degree 3

Yau and Yu [YaYu] wrote a paper that classified the finite subgroups of SL(3,$\mathbb{C}$). They obtained twelve types of finite subgroups of SL(3,$\mathbb{C}$) which are listed below (**A**)-(**L**). It should be noted that the naming conventions presented here for the types of finite subgroups of SL(3,$\mathbb{C}$) and the matrices that generate these subgroups are consistent with their paper.

(A) Diagonal abelian groups. Each element is of the form

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}, \alpha\beta\gamma = 1.$$

(B) Groups isomorphic to transitive linear groups of GL(2,$\mathbb{C}$). The elements of this group have the form

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix}, \alpha(\text{ad - bc}) = 1.$$

(C) Groups generated by (**A**) and T given by

$$T = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix}, \text{ for } \mathrm{P}^{-1}\mathrm{QP} = \mathrm{T},$$

$$\text{where } Q = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & b \\ c & 0 & 0 \end{pmatrix}, \; P = \frac{1}{\sqrt[3]{bc^2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & bc & 0 \\ 0 & 0 & c \end{pmatrix}$$

(D) Groups generated by (**A**), T of (**C**) and

$$R = \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & b \\ 0 & c & 0 \end{pmatrix}, \; abc = -1.$$

(E) Group of order 108 generated by T of (**C**), S, and V given by

$$S^n = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \end{pmatrix}, \; n = 1, 2, 3 \text{ and } \omega = e^{2\pi i/3}$$

$$, \text{ and } V = \frac{1}{\sqrt{-3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}$$

(F) Group of order 216 generated by S, V of (**E**), T of (**C**), and UVU^{-1} given by

$$\mathrm{UVU}^{-1} = \frac{1}{\sqrt{-3}} \begin{pmatrix} 1 & 1 & \omega^2 \\ 1 & \omega & \omega \\ \omega & 1 & \omega \end{pmatrix}, \omega = \mathrm{e}^{2\pi i/3}.$$

(G) Group of order 648 generated by S, V of (**E**), T of (**C**), and U given by

$$U = \begin{pmatrix} \delta & 0 & 0 \\ 0 & \delta & 0 \\ 0 & 0 & \delta\omega \end{pmatrix}, \omega = \mathrm{e}^{2\pi i/3} \text{ and } \delta^3 = \omega^2.$$

(H) Group of order 60 is isomorphic to to the alternating group A_5. It is generated by

$$(12345) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \epsilon^4 & 0 \\ 0 & 0 & \epsilon \end{pmatrix}, (14)(23) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix},$$

$$(12)(34) = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 1 & 1 \\ 2 & s & t \\ 2 & t & s \end{pmatrix},$$

where $\epsilon = \mathrm{e}^{2\pi i/5}$, s = $\epsilon^2 + \epsilon^3 = \frac{1}{2}(-1 - \sqrt{5})$, t = $\epsilon + \epsilon^4 = \frac{1}{2}(-1 + \sqrt{5})$.

(I) Group of order 168 is isomorphic to the permutation group generated by (1234567), (142)(356), and (12)(35). It is generated by

$$(1234567) = \begin{pmatrix} \beta & 0 & 0 \\ 0 & \beta^2 & 0 \\ 0 & 0 & \beta^4 \end{pmatrix}, \; (142)(356) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$(12)(35) = \frac{1}{\sqrt{-7}} \begin{pmatrix} \beta^3 - \beta^4 & \beta^5 - \beta^2 & \beta^6 - \beta \\ \beta^5 - \beta^2 & \beta^6 - \beta & \beta^3 - \beta^4 \\ \beta^6 - \beta & \beta^3 - \beta^4 & \beta^5 - \beta^2 \end{pmatrix},$$

where $\beta = \mathrm{e}^{2\pi i/7}$.

(J) Group of order 180 generated by S of (**E**), and the group (**H**).

(K) Group of order 504 generated by S of (**E**), and the group (**I**).

(L) Group of order 1080 generated by the group (**H**) and W given by

$$\mathrm{W} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & \lambda_1 & \lambda_1 \\ 2\lambda_2 & m & n \\ 2\lambda_2 & n & m \end{pmatrix},$$

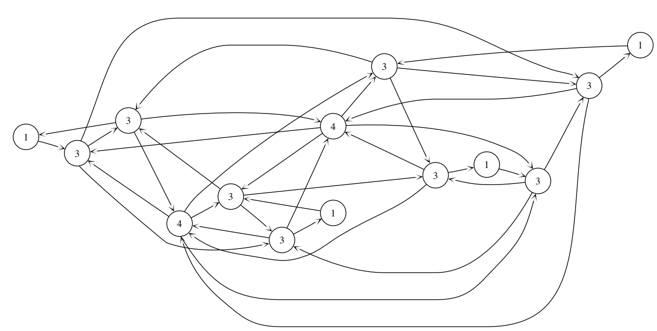
where $\lambda_1 = \frac{1}{4}(-1 + \sqrt{-15})$, $\lambda_2 = \frac{1}{4}(-1 - \sqrt{-15})$,

$$\epsilon = e^{2\pi i/5},\ m = \epsilon^2 + \epsilon^3 = \frac{1}{2}(-1 - \sqrt{5}),\ n = \epsilon + \epsilon^4 = \frac{1}{2}(-1 + \sqrt{5}).$$

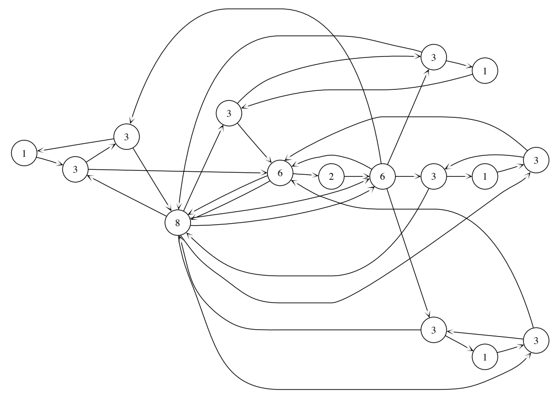
The reader should also be aware that there is a mistake with subgroup (**I**) in the paper published by Yau and Yu [YaYu]. Using the matrices given in the paper, one gets a subgroup of order 336. The problem arises because the matrix for the cycle (12)(35) is incorrect. I modified the matrix for this cycle, and through some calculations found that the matrices presented in this thesis generate a subgroup of order 168 as desired.

Using the mathematical computer package GAP, I was able to compute the M^{c}Kay graphs for the subgroups (**E**)-(**L**).

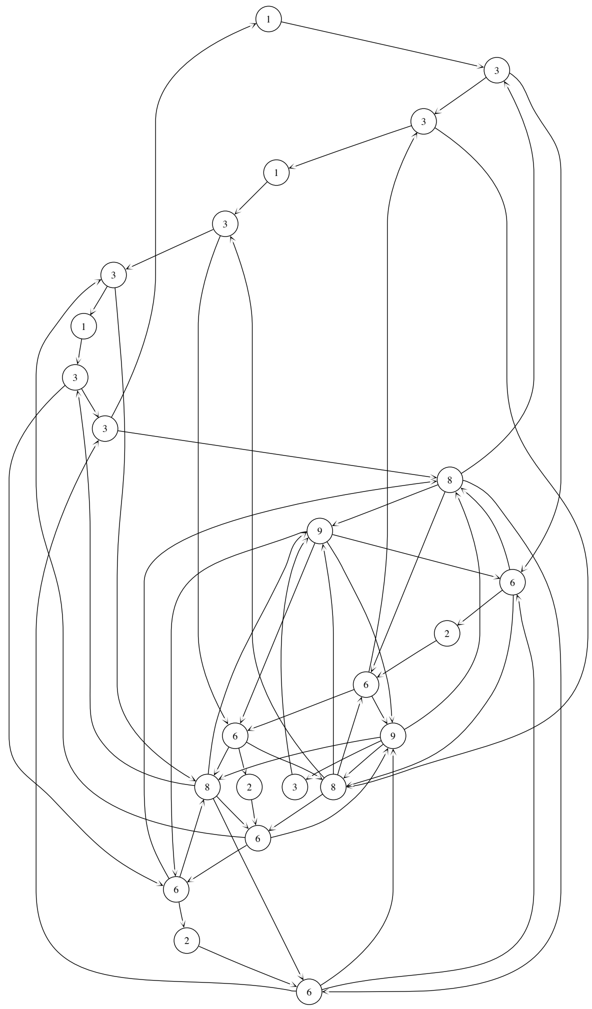
The M^{c}Kay graph for (**E**) :



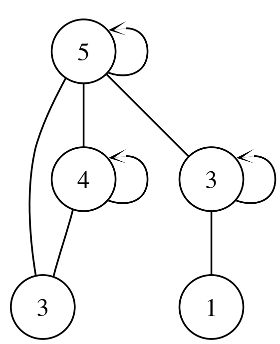
The M^{c}Kay graph for (**F**) :



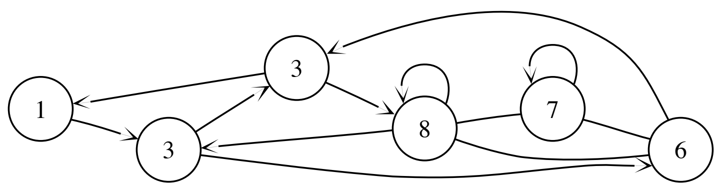
The M^{c}Kay graph for (**G**) :



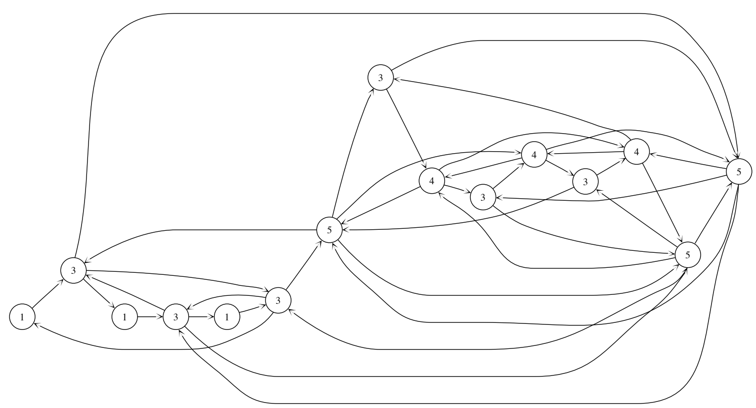
The M^{c}Kay graph for (**H**) :



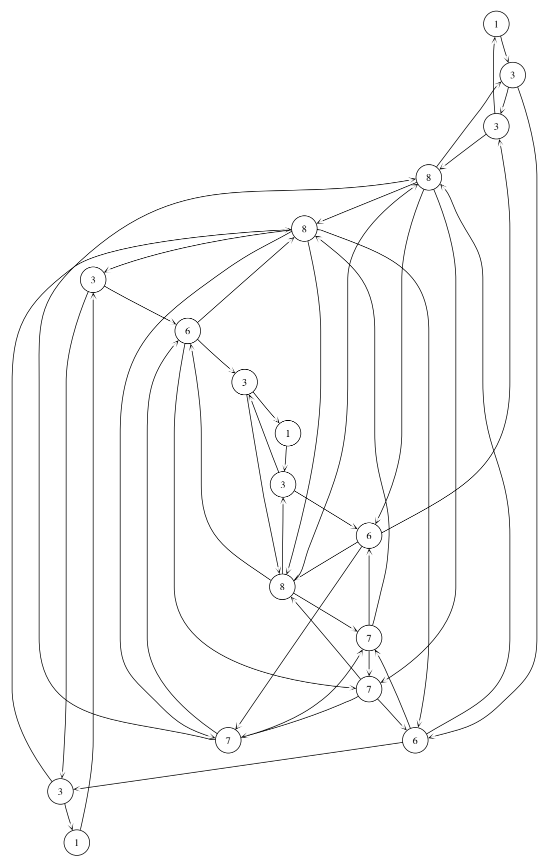
The M^{c}Kay graph for (**I**) :



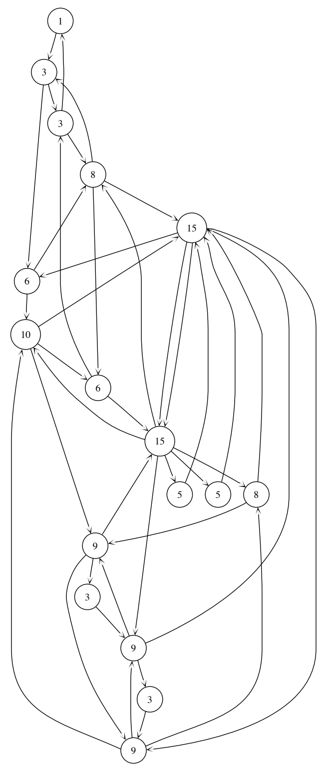
The M^{c}Kay graph for (**J**) :



The M^cKay graph for $(\mathbf{K})$:



The M^{c}Kay graph for (**L**) :



I was unable to produce M^{c}Kay graphs for the subgroups (**A**)-(**D**) by using GAP. The reason for this lies in the fact that these subgroups have free variables as entries of the matrices which allow an infinite number of M^{c}Kay graphs to be generated. However, an adjacency matrix of a M^{c}Kay graph of a reducible representation can be expressed as a sum of adjacency matrices from M^{c}Kay graphs of irreducible representations. This allows us to comment on some of the subgroups of SL(3,$\mathbb{C}$) without using direct computations.

The adjacency matrices of the M^{c}Kay graph for the subgroups (**A**) and (**B**) can be expressed as the sum of an adjacency matrix of the M^{c}Kay graph of G $\subset$ GL(2,$\mathbb{C}$) and an adjacency matrix of the M^{c}Kay graph of G $\subset$ GL(1,$\mathbb{C}$). Finite subgroups of

GL(2,ℂ) are understood from Auslander and Reiten [AuRe] and finite subgroups of GL(1,ℂ) are understood from arguments presented in this thesis. We can say one more thing about the finite subgroups of GL(1,ℂ): they are found by $\frac{1}{det(H)}$ where H is our finite subgroup of GL(2,ℂ).

The M^{c}Kay graphs for the subgroups (**C**) and (**D**) are beyond that scope of this thesis and were not pursued.

2.6 Connected M^{c}Kay Graphs

In the first chapter it is mentioned that a faithful representation is a representation ρ such that $\ker(\rho) = \{1\}$ (*Definition 1.1.6*). This plays an important role in discussing whether a M^{c}Kay graph is connected.

Definition 2.6.1:

A directed graph is called strongly connected if for every pair of vertices u and v, there are paths from u to v and from v to u [West].

Theorem 2.6.2:

If ψ is a faithful representation then the M^{c}Kay graph is strongly connected [McKa].

Before we can prove *Theorem 2.6.2*, we need to introduce Burnside's theorem which can be found in many papers and texts, such as [Farn].

Burnside's Theorem 2.6.3:

Let G be a finite group, ψ a faithful representation, then each irreducible representation is a direct summand of some tensor power $\psi^{\otimes c}$.

Proof of Theorem 2.6.2:

Let ψ be our favorite representation and $\rho_1, \ldots, \rho_r$ be the irreducible representations of G. Suppose ρ_1 is the trivial representation. First, we show that there is a path from ρ_i to ρ_1 for any $\rho_i \in \{\rho_1, ..., \rho_r\}$.

We know that $\psi^{\otimes 2}$ can be expressed as a direct sum of irreducible representations of $\psi \otimes \psi$.

Now, $\psi \otimes \psi = \rho_a \oplus \rho_b \oplus ... \oplus \rho_z$ where $\rho_a, \rho_b, ..., \rho_z \in \{\rho_1, ..., \rho_r\}$ are irreducible representations. So ρ_i has paths from ρ_i to ρ_a, ... , ρ_i to ρ_z.

Consider $\text{mult}_i(\psi^{\otimes 2}) = \text{mult}_i(\psi \otimes \psi) = \text{mult}_i(\rho_a \oplus \rho_b \oplus ... \oplus \rho_z)$. This computes the paths to all vertices of length one away from ρ_i.

Now consider $\text{mult}_i(\psi^{\otimes 3}) = \text{mult}_i(\psi \otimes \psi \otimes \psi) = \text{mult}_i(\{\rho_a \oplus \rho_b \oplus ... \oplus \rho_z\} \otimes \psi) = \text{mult}_i((\rho_a \otimes \psi) \oplus ... \oplus (\rho_z \otimes \psi))$.
This computes the paths to all of the vertices of length two away from ρ_i.

We can continue this argument to some $\text{mult}_i(\psi^{\otimes c})$. By Burnside's theorem, all irreducible representations are in a direct summand of some tensor power $\psi^{\otimes c}$. So we have created a path from ρ_i to ρ_1

We need to show that we can create a path from ρ_i to ρ_j. From the above discussion, we can certainly create a path from ρ_i to ρ_1, where ρ_1 is the trivial representation. So, we just have to show that a path from ρ_1 to ρ_j can be produced to complete the argument.

We know that $\mathrm{mult}_1(\psi^{\otimes d})$ will create paths of length d away from ρ_1. Burnside's theorem says that all irreducible representations are in the direct summand of some tensor power $\psi^{\otimes d}$. But of particular interest, we can get a path from ρ_1 to ρ_j. The result follows. □

Example 2.6.4:

Consider $\mathrm{McK}(D_6, \chi_3)$. $\ker(\chi_3) = \{1\}$. We have seen from *Example 2.1.2* that this M^{c}Kay graph is connected.

If $\ker(\rho)$ contains more group elements than just the identity element, then the M^{c}Kay graph is not connected.

Example 2.6.5:

Consider $McK(D_6, \chi_1)$. By definition, $\ker(\chi_1) = D_6$. We have seen from *Example 2.1.2* that this M^cKay graph is a disjoint union of cycles.

Also we see that each class representative from the character table is in $\ker(\chi_1)$ and we have 3 disjoint cycles in the graph.

We conjecture that there is a relation between the number of class representatives in $\ker(\chi)$ and the number of disjoint cycles in a M^cKay graph.

Example 2.6.6:

Consider McK(D_6, χ_2). ker(χ_2) $= \langle a \rangle$. We have two class representatives in ker(χ_2), namely 1, a. We have seen McK(D_6, χ_2) in *Example 2.1.2* and it did indeed have two disjoint cycles.

Conjecture 2.6.7:

Let ψ be our favorite representation of a group G. The number of class representatives in ker(ψ) equals to the number of disjoint components in McK(G,ψ).

Definition 2.6.8:

The dual of a representation ρ is given by $\rho^* = \overline{\rho}$. A representation is said to be self-dual if $\rho^* = \rho$ [Noor].

Proposition 2.6.9:

Let G be a group, ψ be our favorite representation with λ as its character, and $\rho_1 \ldots \rho_r$ be the irreducible representations of G with characters $\chi_1 \ldots \chi_r$. There are no arrows in the M^{c}Kay graph ($\langle \lambda\chi_i, \chi_j \rangle = \langle \lambda\chi_j, \chi_i \rangle$) if our favorite representation ψ is self-dual [McKa].

Proof:

By definition we know that $m_{ij} = \langle \lambda\chi_i, \chi_j \rangle$ and $m_{ji} = \langle \lambda\chi_j, \chi_i \rangle$. To prove this claim, it amounts to showing that $m_{ij} = m_{ji}$ when ψ is self-dual. We can show this in two steps:

1. If ψ is self-dual, then $m_{ij} = m_{i^*j^*}$.
2. In general, $m_{i^*j^*} = m_{ji}$

We can investigate the first condition.

$m_{i^*j^*} = \langle \lambda\overline{\chi_i}, \overline{\chi_j} \rangle = \overline{\langle \lambda\overline{\chi_i}, \overline{\chi_j} \rangle} = \langle \overline{\lambda}\chi_i, \chi_j \rangle = \langle \lambda\chi_i, \chi_j \rangle$ =m$_{ij}$. This is true since ψ is self-dual.

We can now show the second condition.

$$m_{i^*j^*} = \langle \lambda\overline{\chi_i}, \overline{\chi_j} \rangle = \tfrac{1}{|G|} \sum_{g \in G} \lambda(g)\overline{\chi_i(g)\overline{\chi_j(g)}} = \tfrac{1}{|G|} \sum_{g \in G} \lambda(g)\chi_j(g)\overline{\chi_i(g)} = \langle \lambda\chi_j, \chi_i \rangle = m_{ji}.$$

We have shown both of the conditions, so by combining them we have $m_{ij} = m_{i^*j^*}$ $= m_{ji}$ when ψ is self-dual. □

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